On the integer solutions of the Pell equation \( x^2 - 18y^2 = 4^k \)

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ABSTRACT: The binary quadratic diophantine equation represented by \( x^2 - 18y^2 = 4^k \), \( k > 0 \) is considered. A method for obtaining infinitely many non-zero distinct integer solutions of the Pell equation considered above is illustrated. A few interesting relations among the solutions and special figurate numbers are presented. Recurrence relations on the solutions are given.

KEYWORDS - Pell equation, binary quadratic diophantine equation, integer solutions.

I. INTRODUCTION

It is well known that the Pell equation \( x^2 - Dy^2 = 1 \) \( (D > 0 \) and square free) has always positive integer solutions. When \( N \neq 1 \), the Pell equation \( x^2 - Dy^2 = N \) may not have any positive integer solutions. For example, the equations \( x^2 - 3y^2 = 1 \) and \( x^2 - 7y^2 = 1 \) have no integer solutions. When \( k \) is a positive integer and \( D \in (k^2 \pm 1, k^2 \pm 2) \), positive integer solutions of the equations \( x^2 - Dy^2 = \pm 4 \) and \( x^2 - Dy^2 = \pm 1 \) have been investigated by Jones in [1]. In [2-11], some specific Pell equations and their integer solutions are considered. In [12], the integer solutions of the Pell equation \( x^2 - (k^2 + k)y^2 = 2^k \) have been considered. In [13], the Pell equation \( x^2 - (k^2 - k)y^2 = 2^k \) is analysed for the integer solutions.

This communication concerns with the Pell equation \( x^2 - 18y^2 = 4^k \), \( k > 0 \) and infinitely many positive integer solutions are obtained. A few interesting relations among the solutions and special figurate numbers are presented. Recurrence relations on the solutions are given.

II. Notations

- \( \tau_{m,n} \): Polygonal number of rank \( n \) with sides \( m \)
- \( P_m^n \): Pyramidal number of rank \( n \) with sides \( m \)
- \( CP_m^n \): Centered Pyramidal number of rank \( n \) with sides \( m \)
- \( FC_m^n \): Prism number of rank \( n \) with sides \( m \)
- \( \sigma(x) \): Gnomonic number
- \( SO(n) \): Stella octangula number
- \( CD(n) \): Centered Dodecahedral number
- \( CC(n) \): Centered Cube number
- \( TOH(n) \): Truncated Octahedral number
- \( PTP(n) \): Pentatope number
- \( HO(n) \): Huay Octahedral number
- \( N_d(n) \): \( n^a \)-dimensional nexus number
- \( g_{m,n} \): \( m \)-gram number of rank \( n \)
- \( RD(n) \): Rhombic Dodecahedral number
Method of ANALYSIS

The Pell equation to be solved is $x^2 - 13y^2 = 4^k$ (1)

Let $(X_0, Y_0)$ be the initial solution of (1) which is given by

$$X_0 = 17.2^k ; Y_0 = 2^k + 2 , \ k \in \mathbb{Z} - \{0\}$$

To find the other solutions of (1), consider the Pellian equation

$$x^2 = 13y^2 + 1$$

whose general solution $(x_n, y_n)$ is given by

$$x_n = \frac{1}{2} f_n$$

$$y_n = \frac{1}{\sqrt{3}} g_n$$

where

$$f_n = \left(17 + 12\sqrt{2}\right)^{n+1} + \left(17 - 12\sqrt{2}\right)^{n+1}$$

$$g_n = \left(17 + 12\sqrt{2}\right)^{n+1} - \left(17 - 12\sqrt{2}\right)^{n+1} , \ n = 0,1,2,\ldots$$

Applying Brahmagupta lemma between $(X_0, Y_0)$ and $(x_n, y_n)$, the sequence of non-zero distinct integer to (1) are obtained as

$$X_{n+1} = 2^{k+1} (x_n + 12\sqrt{2} g_n)$$

$$Y_{n+1} = 2^{k+1} (12\sqrt{2} x_n + 17 g_n)$$

The recurrence relations satisfied by the solutions of (1) are given by

$$X_{n+1} = 34X_{n-1} + X_{n+1} = 0 , \ X_1 = 577.2^k , X_2 = 1960.1.2^k$$

$$Y_{n+1} = 34Y_{n-1} + Y_{n+1} = 0 , \ Y_1 = 17.2^{k+1} , Y_2 = 1155.2^{k+2}$$

From (2) and (3), the values of $f_n$ and $g_n$ are found to be

$$f_n = \frac{1}{2^{k+1}} (34X_{n+1} - 144Y_{n+1})$$

$$g_n = \frac{1}{2^{k+2}} (204Y_{n+1} - 48X_{n+1})$$

In view of (4), the following relations are observed

1. $24(34X_{n+1} - 144Y_{n+1})^2 - 12(204Y_{n+1} - 48X_{n+1})^2$ is a nasty number.
2. $X_{n+1} = 17X_{n+1} + 72Y_{n+1}$
3. $X_{n+1} = 577X_{n-1} + 2448Y_{n+1}$
4. $Y_{n+1} = 4X_{n+1} + 17Y_{n+1}$
5. $Y_{n+1} = 13X_{n+1} + 577Y_{n+1}$
6. $Y_{n+1} = Y_{n-1} + 8X_{n-2}$
7. $Y_{n+1} = 17Y_{n-1} - 4X_{n+2}$
8. $34X_{2n+2} - 144Y_{2n+1} + 2^{k+1} \equiv 0(\mod 2^k)$
9. $17X_{2n+2} - 72Y_{2n+1} - f_2(f_3)2^{k-1} + t_6z_12^{k-1} + G(f_n)2^{k-1} \equiv 0(\mod 2^k)$
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10. $34X_{2n+2} - 144Y_{2n+2} = (CP_{fn}^2 - 2C P_{fn}^2), 2^{k+1}$
11. When $k \equiv 0 (mod \ 3), 34X_{2n+1} - 144Y_{2n+1} + 3(34X_{n+1} - 144Y_{n+1})$ is a cubic integer.
12. $34X_{2n+1} - 144Y_{2n+1} - P_{fn}^2 \cdot 2^{k+1} + 3t_{2,1}x_{fn}, 2^{k+1} \equiv 0 (mod \ f_n)$
13. $34X_{2n+2} - 144Y_{2n+2} - P_{fn}^2, 2^{k+1} + 3H(\frac{f_n}{2}), 2^{k+1} + t_{12,1}x_{fn}, 2^{k} - 3P_{fn}^2 2^{k+1} \equiv 0 (mod \ 3)$
14. $17X_{2n+1} - 72Y_{2n+2} - 5O(f_n), 2^{k-1} + CP_{fn}^2, 2^{k} \equiv 0 (mod \ f_n)$
15. $17X_{2n+1} - 72Y_{2n+2} - 5O(f_n), 2^{k} + (G(f_n), 2^{k} \equiv 0 (mod \ 2^k)$
16. $34X_{2n+1} - 144Y_{2n+1} - TOH(f_n), 2^{k} + 3CP_{fn}^2, 2^{k+1} - 3t_{18,1}x_{fn}, 2^{k} \equiv 0 (mod \ 6)$
17. $5(34X_{2n+2} - 144Y_{2n+2}) = 2^{k+1} + N_4(f_n), 2^{k} - CD(f_n), 2^{k} - 13t_{14,1}x_{fn}, 2^{k} - t_{2,1}x_{fn}, 2^{k+2}$
18. $34X_{3n+4} - 144Y_{3n+4} = 3TP(f_n), 2^{k+2} - CC(f_n), 2^{k} + RD(f_n), 2^{k} + 11t_{14,1}x_{fn}, 2^{k} + 3t_{2,1}x_{fn}, 2^{k+1}$
19. When $k \equiv 0 (mod \ 4), 34X_{4n+4} - 144Y_{4n+4} + t_{4,1}x_{fn}, 2^{k+2} - 2^{k+1} is a biquadratic integer.
20. Define $X = 34X_{n+1} - 144Y_{n+1}$ and $Y = 204Y_{n+1} - 48X_{n+1}$. Note that $(X, Y)$ satisfies the hyperbola $Y^2 = 2X^2 - 32$.

III. CONCLUSION

To conclude, one may search for other patterns of solutions to the similar equation considered above.

REFERENCES