International Journal of Engineering Science Invention (IJESI)
ISSN (Online): 2319-6734, ISSN (Print): 2319-6726
www.ijesi.org ||Volume 12 Issue 3 March 2023 || PP 45-52

Special Concircular Lie-Recurrence in a Finsler space equipped with non-symmetric connection

[1]S.K. Tiwari, [2]Ved Mani and [3]C.P. Maurya
[1,2]Department of Mathematics
K.S. Saket P.G. College, Ayodhya (224123)
sktiwarisaket@yahoo.com; ved27051986@gmail.com
[3]Adarsh Inter College Saltauwa Gopalpur, Basti (272190)

Abstract
In present paper we deals with Lie-recurrence generated by a concircular vector field in a Finsler space equipped with non-symmetric connection. In this communication, we have observed that in a Finsler space equipped with non symmetric connection admitting concircular Lie-recurrence. In the later sections of the communication results have been derived in a recurrent Finsler space of second order with non-symmetric connection. In last section we studies a symmetric Finsler space of second order with non-symmetric connection.

Date of Submission: 14-03-2023 Date of Acceptance: 31-03-2023

I. Introduction

We consider n-dimensional Finsler space F^n [10] having 2n-line element $(x^i, \dot{x}^j)$ $(i, j = 1, 2, ..., n)$ equipped with non-symmetric connection Γ^i_{jk}. The non-symmetric connection Γ^i_{jk} is based on non-symmetric fundamental metric tensor $g_{ij}(x, \dot{x}) \neq g_{ji}(x, \dot{x})$. Nitescu (1974) [5] defined non-symmetric connection Γ^i_{jk} as follows :

(1.1) (a) $\Gamma^i_{jk} = M^i_{jk}(x, \dot{x}) + \frac{1}{2} N^i_{jk}(x, \dot{x})$

One more connection parameter $\hat{\Gamma}^i_{jk}$ has been introduced by Pandey and Gupta [6].

(1.1) (b) $\hat{\Gamma}^i_{jk}(x, \dot{x}) = M^i_{jk}(x, \dot{x}) - \frac{1}{2} N^i_{jk}(x, \dot{x})$

The covariant derivative of the tensor field $T^i_j(x, \dot{x})$ with respect ot x^k is defined Pande H.D. and Tiwari, S.K. [7] in two distinct ways as :

(1.2) (a) $T^i_{\overset{+}{j|k}} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\Gamma^m_{pk}\dot{x}^p + T^m_j \Gamma^i_{mk} - T^i_m \Gamma^m_{jk}$

(1.2) (b) $T^i_{\overset{-}{j|k}} = \partial_k T^i_j - (\dot{\partial}_m T^i_j)\hat{\Gamma}^m_{pk}\dot{x}^p + T^m_j \hat{\Gamma}^i_{mk} - T^i_m \hat{\Gamma}^m_{jk}$

where $\hat{\Gamma}^i_{jk} = \Gamma^i_{kj}$

The commutation formula involving the covariant derivative defined in (1.2) is given by [7].

(1.3) $T^i_{\overset{+}{j|hk}} - T^i_{\overset{+}{j|kh}} = -(\dot{\partial}_m T^i_j) R^m_{shk}\dot{x}^s + T^m_j R^i_{mhk} - T^i_m R^m_{jhk} + T^i_{\overset{+}{j|m}} N^m_{kh}$

where

(1.4) $R^h_{ijk} = \partial_k \Gamma^h_{ij} - \partial_j \Gamma^h_{ik} + (\dot{\partial}_m \Gamma^h_{ik})\Gamma^m_{sj}\dot{x}^s - (\dot{\partial}_m \Gamma^h_{ij})\Gamma^m_{sk}\dot{x}^s + \Gamma^p_{ij}\Gamma^h_{pk} - \Gamma^p_{ik}\Gamma^h_{pj}$

is a curvature tensor.

In view of covariant derivative we have the following commutation formula for tensor T^i_j is given by [6] :

(1.5) $\dot{\partial}_k \left(T^i_{\overset{+}{j|h}}\right) - (\dot{\partial}_k T^i_j)_{\overset{+}{|h}} = T^m_j \Gamma^i_{mkh} - T^i_m \Gamma^m_{kjh} - (\dot{\partial}_m T^i_j)\Gamma^m_{kph}\dot{x}^p$

where $\Gamma^i_{mkh} = \dot{\partial}_m \Gamma^i_{kh}$.

We shall extensively use the following identities, notations and contractions;

(1.6) (a) $x^i_{\overset{+}{|}k} = 0$

(b) $x^i_{\overset{-}{|}k} = 0$

(c) $R^i_{jk} = R^i_{hjk}\dot{x}^h$

(d) $R^i_j = R^i_{hj}\dot{x}^h$

(e) $R^i_{hjk} = -R^i_{hkj}$

(f) $R^i_i = (n-1)R$

and

(g) $N^i_{jk} = -N^i_{kj} = \Gamma^i_{jk} - \Gamma^i_{kj}$

The Lie-derivative of an arbitrary tensor $T^i_j(x,\dot{x})$ and non-symmetric connection Γ^i_{jk} are expressible in the forms given as under [7]

(1.7) (a) $L_v T^i_j(x,\dot{x}) = T^i_{j\overset{+}{|}k} v^k + \left(\dot{\partial}_k T^i_j\right) v^k_{\overset{+}{|}h}\dot{x}^h - T^k_j v^i_{\overset{+}{|}k} + T^i_k v^k_{\overset{+}{|}j}$

(b) $L_v R^i_{hjk} = R^i_{hjk\overset{+}{|}s} v^s + R^i_{sjk} v^s_{\overset{-}{|}h} + R^i_{hsk} v^s_{\overset{-}{|}j} + R^i_{hjs} v^s_{\overset{-}{|}k} - R^s_{hjk} v^i_{\overset{-}{|}s} + \left(\dot{\partial}_s R^i_{hjk}\right) v^s_{\overset{-}{|}r}\dot{x}^r$

(1.8) $L_v \Gamma^i_{jk}(x,\dot{x}) = v^i_{\overset{+}{|}jk} - R^i_{jkh} v^h + \left(\dot{\partial}_s \Gamma^i_{jk}\right) v^s_{\overset{+}{|}h}\dot{x}^h$

We have the following commutation formula [7].

(1.9) $L_v\left(\dot{\partial}_k T^i_j\right) - \dot{\partial}_k\left(L_v T^i_j\right) = 0$

(1.10) $L_v\left(T^i_{j\overset{+}{|}k}\right) - \left(L_v T^i_j\right)_{\overset{+}{|}k} = T^h_j L_v \Gamma^i_{hk} - T^i_h L_v \Gamma^h_{jk} + \left(\dot{\partial}_h T^i_j\right)\left(L_v \Gamma^h_{sk}\right)\dot{x}^s$

and

(1.11) $L_v\left(\Gamma^i_{hj\overset{+}{|}k}\right) - \left(L_v \Gamma^i_{hj}\right)_{\overset{+}{|}k} = L_v R^i_{hjk} + \Gamma^i_{rhj}\left(L_v \Gamma^r_{lk}\right)\dot{x}^l - \Gamma^i_{rhk}\left(L_v \Gamma^r_{lj}\right)\dot{x}^l + N^r_{kj}\left(L_v \Gamma^i_{hr}\right)$

We now consider an infinitesimal transformation given as :

(1.12) $\bar{x}^i = x^i + \in v^i$

Such a transformation is generated by a contravariant vector v^i which depends on positional co-ordinates only, the term $\in$ is an infinitesimal constant.

2. Special Con-circular Lie-recurrence in a Finsler space F_n^* :

In the Finsler space F_n^* under consideration we assume that the space is admitting an infinitesimal transformation given by (1.12), which is generated by a special concircular vector field to be characterised by

(2.1) (a) $v^i_{\overset{+}{|}k} = \rho\delta^i_j$

(b) $v^i_{\overset{-}{|}k} = \rho\delta^i_j$

where ρ is not a constant.

We now allow a partial differentiation in (2.1) with respect to $\dot{x}^k$, we get

(2.2) $\dot{\partial}_k\left(v^i_{\overset{+}{|}j}\right) = \left(\dot{\partial}_k \rho\right)\delta^i_j$

We now take into account the commutation formula given by (1.5) in (2.2), we get

(2.3) $\dot{\partial}_k\left(v^i_{|j}\right)-\left(\dot{\partial}_k v^i\right)_{|j}=v^m\Gamma^i_{mkj}-\left(\dot{\partial}_m v^i\right)\Gamma^m_{kpj}\dot{x}^p$

But, since v^i is a function of positional co-ordinates only hence from (2.3), we get

(2.4) $\dot{\partial}_k\left(v^i_{|j}\right)=v^m\Gamma^i_{mkj}$

Using (2.1) in (2.4), we get

(2.5) $\left(\dot{\partial}_k\rho\right)\delta^i_j=v^m\Gamma^i_{mkj}$

We now make the supposition that the special concircular transformation given by (1.12) has a Lie-recurrence in the Finsler space F^*_n. i.e.

(2.6) $L_v R^i_{jkh}=\phi R^i_{jkh}$

Where ϕ is a non-zero scalar.

It can easily be verified that the scalar ϕ appearing in (2.6) is a functions of positional co-ordinates only.

In view of (2.1) (b), (1.7) (b), (2.6) may be expressed in the following form:

(2.7) $R^i_{hjk|s}v^s=(\phi-2\rho)R^i_{hjk}$

Where, we have taken into account the fact that R^i_{hjk} is homogeneous of degree zero in its directional arguments.

Differentiating (2.1), $\oplus$-covarintly in the sense of (1.2), we get

(2.8) $v^i_{|jk}=\rho_k\delta^i_j$ where $\rho_k=\rho_{|k}$

Applying commutation in (2.8) with respect to indices j and k, we get

(2.9) $v^i_{|jk}-v^i_{|kj}=\rho_k\delta^i_j-\rho_j\delta^i_k$

Applying commutation formula given (1.3) in (2.9), we get

(2.10) $-\left(\dot{\partial}_m v^i\right)R^m_{sjk}\dot{x}^s+v^m R^i_{mjk}=\rho_k\delta^i_j-\rho_j\delta^i_k-v^i_{|m}N^m_{jk}$

Using (1.6) in (2.10), we get

(2.11) $-\left(\dot{\partial}_m v^i\right)R^m_{sjk}\dot{x}^s+v^m R^i_{mjk}+v^i_{|m}\left(\Gamma^m_{jk}-\Gamma^m_{kj}\right)=\rho_k\delta^i_j-\rho_j\delta^i_k$

Again using (2.1) and the fact the contravariant vector v^i is independent of directional arguments in (2.11), we get

(2.12) $v^m R^i_{mjk}+\rho\left(\Gamma^i_{jk}-\Gamma^i_{kj}\right)=\rho_k\delta^i_j-\rho_j\delta^i_k$

We allow a contraction in (2.12) with respect to the indices i and j, we get

(2.13) $\rho_k=\frac{1}{(n-1)}\left[v^m R^i_{mik}+\rho\left(\Gamma^i_{ik}-\Gamma^i_{ki}\right)\right]$

Using (2.13) in (2.12), we get

(2.14) $v^m R^i_{mjk} + \rho\left(\Gamma^i_{jk} - \Gamma^i_{kj}\right) = \frac{1}{(n-1)}\left[v^m R^p_{mpk} + \rho\left(\Gamma^p_{pk} - \Gamma^p_{kp}\right)\right]\delta^i_j - \frac{1}{(n-1)}\left[v^m R^p_{mpj} + \rho\left(\Gamma^p_{pj} - \Gamma^p_{jp}\right)\right]\delta^i_k$

As a special case, if we assume that the connection coefficient Γ^i_{jk} appearing in (2.13) is symmetric one then from (2.13), we shall get

(2.15) $\rho_k = \frac{1}{(n-1)} v^m R^i_{mik}$

Using (2.15) in (2.12), we get

(2.16) $v^m\left[(n-1)R^i_{mjk} + R_{mj}\delta^i_k - R_{mk}\delta^i_j\right] = 0$

Where we have written $R^i_{mij} = R_{mj}$.

We now take into account the Lie-derivative of ρ_k and in accordance with (1.7), we write it as

(2.17) $L_v \rho_k = \rho_{k\overset{+}{|}h} v^h + \left(\dot{\partial}_h \rho_k\right)\rho^h_{|l}\dot{x}^l + \rho_h v^h_{\overset{+}{|}k}$

We use (2.1) in (2.17), we get

(2.18) $L_v \rho_k = \rho_{k\overset{+}{|}h} v^h + \rho\left(\dot{\partial}_h \rho_k\right)\dot{x}^h + \rho\,\rho_k$

Since ρ_k has been assumed to be degree zero is its directional arguments hence from (2.18), we get

(2.19) $L_v \rho_k = \rho_{k\overset{+}{|}h} v^h + \rho\,\rho_k$

We now allow a transformation to (2.7) by v^h and get

(2.20) $R^i_{hjk\overset{+}{|}s} v^s v^h = (\phi - 2\rho) R^i_{hjk} v^h$

We now differentiate (2.12), $\oplus$-covariantly in sense of (1.2) with respect to x^l, we get

(2.21) $v^m_{|l} R^i_{mjk} + v^m R^i_{mjk\overset{+}{|}l} + \rho_{|l}\left(\Gamma^i_{jk} - \Gamma^i_{kj}\right) + \rho\left(\Gamma^i_{jk} - \Gamma^i_{kj}\right)_{|l} = \rho_{k\overset{+}{|}l}\delta^i_j - \rho_{j\overset{+}{|}l}\delta^i_k$

We now allow a transvection in (2.21) by v^l and there after using (2.1), we get

(2.22) $\rho v^m R^i_{mjk} + v^m v^l R^i_{mjk\overset{+}{|}l} + v^l \rho_l N^i_{jk} + \rho v^l\left[\left(\partial_l N^i_{jk}\right) - \left(\dot{\partial}_m N^i_{jk}\right)\Gamma^m_{sl}\dot{x}^s + N^m_{jk}\Gamma^i_{ml} - N^i_{mk}\Gamma^m_{jl} - N^i_{jm}\Gamma^m_{kl}\right]$

$$= v^l\left(\rho_{k\overset{+}{|}l}\delta^i_j - \rho_{j\overset{+}{|}l}\delta^i_k\right)$$

Using (2.12) and (2.20) in (2.22), we get

(2.23) $\rho\left[\rho_k\delta^i_j - \rho_j\delta^i_k - \rho N^i_{jk}\right] + (\phi - 2\rho)\left[\rho_k\delta^i_j - \rho_j\delta^i_k - \rho N^i_{jk}\right]$

$$+\rho v^l\left[\left(\partial_l N^i_{jk}\right) - \left(\dot{\partial}_m N^i_{jk}\right)\Gamma^m_{sl}\dot{x}^s + N^m_{jk}\Gamma^i_{ml} - N^i_{mk}\Gamma^m_{jl} - N^i_{jm}\Gamma^m_{kl}\right]$$

$$= v^l\left[\rho_{k\overset{+}{|}l}\delta^i_j - \rho_{j\overset{+}{|}l}\delta^i_k\right]$$

We now allow a contraction in (2.23) with respect to the indices i and j, we get

(2.24) $$(n-1)(\phi-\rho)\rho_k - \rho(\phi-\rho)N^i_{ik} + \rho v^l\left[\left(\partial_l N^i_{ik}\right) - \left(\dot{\partial}_m N^i_{ik}\right)\Gamma^m_{sl}\dot{x}^s + N^m_{ik}\Gamma^i_{ml} - N^i_{mk}\Gamma^m_{il} - N^i_{im}\Gamma^m_{kl}\right] = (n-1)v^l \rho_{k\overset{+}{|}l}$$

Making use of (2.24) in (2.19) will not enable us to state about the Lie-recurrence of the covariant derivative of the scalar ρ and therefore, we can state:

Theorem (2.1) : In a Finsler space F_n^* equipped with non-symmetric connection and admitting concircular Lie-recurrence respectively characerised by (2.1) and (2.6) covariant derivative of the scalar ρ is not Lie-recurrent.

But however if we assume that the connection coefficient Γ^i_{jk} of Finsler space F_n^* is symmetric one in its lower indices j and k, then as a result of this assumption, we immediately get the following from (2.24).

(2.25) $$(\phi-\rho)\rho_k = v^l \rho_{k\overset{+}{|}l}$$

Using (2.19) in (2.25), we get

(2.26) $$L_v \rho_k = \phi\ \rho_k$$

Therefore in such a special case, we can state :

Corollary (2.1) :

In a Finsler space F_n^* admitting concircular Lie-recurrence respectively characterised by (2.1) and (2.6), the covariant derivative of the scalar ρ appearing in (2.1) is Lie-recurrent with respect to the Lie-recurrence provided the connection coefficient Γ^i_{jk} of Finsler space F_n^* be assumed to be symmetric.

3. Recurrent Finsler space F_n^* of second order and special concircular Lie-recurrence :

Definition (3.1) : The Finsler space F_n^* equipped with non-symmetric connection is said to be recurrent of second order if

(3.1) $$R^i_{jkh\overset{+}{|}lm} = \beta_{lm} R^i_{jkh}, \qquad R^i_{jkh} \neq 0$$

Where β_{lm} $(x, \dot{x})$ are the component of a non-zero recurrence covariant tensor of second order.

Differentiating (2.7) $\oplus$-covariantly with respect to x^m, we get

(3.2) $$R^i_{jkh\overset{+}{|}lm} v^l + R^i_{jkh\overset{+}{|}l} v^l_{\overset{+}{|}m} = (\phi_m - 2\rho_m) R^i_{jkh} + (\phi - 2\rho) R^i_{jkh\overset{+}{|}m}$$

Where $\phi_m = \phi_{\overset{+}{|}m}$ and $\rho_m = \rho_{\overset{+}{|}m}$.

Introducing (2.1) and (3.1) in (3.2), we get

(3.3) $$(\phi - 3\rho) R^i_{jkh\overset{+}{|}m} = \left(v^l \beta_{lm} - \phi_m + 2\rho_m\right) R^i_{jkh}$$

In the definition (3.1) it has been assumed that $R^i_{jkh} \neq 0$, which will mean that the recurrent Finsler space F_n^* of second order is necessary non-flat and also the recurrence covariant tensor β_{lm} of second order is non-symmetric because, if it be assumed that such a space is symmetric then $R^i_{jkh|m}=0$ will obviously imply $R^i_{jkh|lm}=0$ and as such from (3.1) we shall automatically get $\beta_{lm}=0$ which will lead to a contradiction of definition (3.1). In the light of these observations equation (3.3) will imply either of the following two conditions :

(i) $\phi-3\rho=0, v^l\beta_{lm}-\phi_m+2\rho_m=0$

(ii) $\phi-3\rho\neq 0, v^l\beta_{lm}-\phi_m+2\rho_m\neq 0$

In the light of the second observation, we can write (3.3) in the following alternative from

(3.4) $R^i_{jkh|m}=\gamma_m R^i_{jkh}$

where $\gamma_m=\dfrac{v^l\beta_{lm}-\phi_m+2\rho_m}{\phi-3\rho}$

Equation (3.4) automatically tells that the Finsler space F_n^* equipped with non-symmetric connection is recurrent of order one but it has been observe in [8] that a recurrent Finsler space F_n^* does not admit a con-circular vector field and therefore under this observation we easily arrive at the conclusion that a recurrent Finsler space F_n^* will not admit a special concircular Lie-recurrence and therefore we can state :

Theorem (3.1). A bi-recurrent Finsler space F_n^* of second order with special concircular Lie-recurrence necessarily satisfies :

(3.5) (i) $\phi-3\rho=0$

(ii) $v^l\beta_{lm}-\phi_m+2\rho_m=0$

Communicating (3.1) with respect to the indices l and m, we get

(3.6) $R^i_{jkh|lm}-R^i_{jkh|ml}=(\beta_{lm}-\beta_{ml})R^i_{jkh}$

Using (1.3) in (3.6), we get

(3.7) $$-\left(\dot\partial_p R^i_{jkh}\right)R^p_{slm}\dot x^s+R^p_{jkh}R^i_{plm}-R^i_{pkh}R^p_{jlm}-R^i_{jph}R^p_{klm}-R^i_{jkp}R^p_{hlm}$$
$$+R^i_{jkh|p}N^p_{lm}=(\beta_{lm}-\beta_{ml})R^i_{jkh}=B_{lm}R^i_{jkh}$$

where $B_{lm}=\beta_{lm}-\beta_{ml}$.

Taking the Lie-derivative of both sides of (3.7) and there after using (2.6), we get

(3.8) $$-L_v\left(\dot\partial_p R^i_{jkh}\right)R^p_{slm}\dot x^s-\phi\left(\dot\partial_p R^i_{jkh}\right)R^p_{slm}\dot x^s+2\phi R^p_{jkh}R^i_{plm}-2\phi R^i_{pkh}R^p_{jlm}$$
$$-2\phi R^i_{jph}R^p_{klm}-2\phi R^i_{jkp}R^p_{hlm}+\left(L_v R^i_{jkh|p}\right)N^p_{lm}+R^i_{jkh|p}L_v\left(N^p_{lm}\right)$$
$$=\left(L_v B_{lm}\right)R^i_{jkh}+\phi\, B_{lm}R^i_{jkh}$$

In the light of (3.8), we can therefore state

Theorem (3.2).

In a recurrent Finsler space F_n^* of second order the skew-symmetric part of the recurrence tensor $\beta_{lm}(x,\dot{x})$ appearing in (3.1) with special concircular Lie-recurrence is not Lie-recurrent in general. But however if we assume that

$$(3.9)\quad -L_v\left(\dot{\partial}_p R^i_{jkh}\right)R^p_{slm}\dot{x}^s - \phi\left(\dot{\partial}_p R^i_{jkh}\right)R^p_{slm}\dot{x}^s + 2\phi R^p_{jkh}R^i_{plm} - 2\phi R^i_{pkh}R^p_{jlm}$$
$$-2\phi R^i_{jph}R^p_{klm} - 2\phi R^i_{jkp}R^p_{hlm} + \left(L_v R^i_{jkh\overset{+}{|}p}\right)N^p_{lm} + R^i_{jkh\overset{+}{|}p}\left(L_v N^p_{lm}\right) = 0$$

Then using (3.9) in (3.8), we immediately get

$$(3.10)\quad L_v B_{lm} = \psi B_{lm}$$

where $\psi = -\phi$

Therefore in the light of (3.10), we can state the following :

Corollary (3.1) : In a recurrenct Finsler space F_n^* of second order, the skew-symmetric part of the recurrence tensor $\beta_{lm}(x,\dot{x})$ with special concircular Lie-recurrence is Lie-recurrent with respect to the Lie-recurrence provided (3.9) holds.

Referrence :

[1] Duggle, K.I. : Curvature inheriting symmetry in Riemannian space with application of Fluid space time, J. Math Phys., 33(9), (1992), 2989-2996.

[2] Gupta, K.K. : Certain investigation in the differential geometry of Finsler spaces with non-symmetric connection, Ph.D. Thesis, University of Gorakhpur (India), 1980.

[3] Mishra, R.B. : A symmetric Finsler space, Tensor (N.S.) 24, (1972), 346-350.

[4] Mishra, C.K. and Lodhi, Gautam : On curvature inheriting symmetry in Finsler space, Acta Universities Apulensis, 30 (2012), 39-48.

[5] Nitsescu : Cataline Bianchies identities in a non-symmetric connection space. Bul. Inst. Politenn. Iasi (N.S.) 20(24) Fasc. 1-2, Sec. I, 69-72 (1974)

[6] Pandey, H.D. and Gupta, K.K. : Some special properties of R^h_{ijk} and Q^h_{ijk} in recurrent Finsler spaces. Indian J. Pure applied Math. 10(8) : 1020-1030 (1979).

[7] Pandey, H.D. and Tiwari, S.K. : Existence of affine motion in a $R-\oplus$ Recurrent Finsler space : J. Nat. Acad. Math. Vol. (10), 1992-1996.

[8] Pande, P.N. : A recurrent Finsler space with concircular vector field, Acta Math. Acad. Sci. (Hungarica) 35 (1980), 465-466.

[9] Pandey, Vaishali and Pande, P.N. On a special concircular Lie-recurrence in Finsler space, luter J. of pure and applied Math. Vol. 92, No. 3 (2014), 359-367.

[10] Rund, H. The differential geometry of Finsler spaces, Springer Verlag, Berlin (1959).

S.K. Tiwari. "Special Concircular Lie-Recurrence in a Finsler space equipped with non-symmetric connection." *International Journal of Engineering Science Invention (IJESI)*, Vol. 12(3), 2023, PP 45-52. Journal DOI- 10.35629/6734