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Weighted composition operators from α-Bloch spaces to Bers-Orlicz spaces

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ABSTRACT : *In this paper we use Young's function to define the Bers-Orliicz space as a generalization of Bers space, a space consists of analytic functions. Moreover, the boundedness and compactness of the weighted composition operators from α-Bloch space to Bers-Orlicz space on the unit open disk are characterized.*

KEYWORDS - *α-Bloch space, Bers-orlicz space, boundedness, compactness, Weighted composition operators*

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I.INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$.

Definition 1.1: Let $\mu: \mathbb{D} \to (0,1)$ is a radial weight function, that is, μ is a continuous function which is monotonically decreasing in a neighborhood of 1, and $\mu = \mu(|z|)$, $\lim_{z\to 1^-} \mu(z) = 0$. The definitions of the Bloch space $\mathscr{B}_\mu$ and the little Bloch space $\mathscr{B}_{\mu,0}$ on $\mathbb{D}$ are:

$$\mathscr{B}_\mu = \left\{ f(z) \in H(\mathbb{D}) : \|f\|_{\mathscr{B}_\mu} = f(0) + \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty \right\} \quad (1)$$

$$\mathscr{B}_{\mu,0} = \left\{ f(z) \in \mathscr{B}_\mu : \lim_{z\to 1^-} \mu(z)\left|f'(z)\right| = 0 \right\} \quad (2)$$

Where $\mathscr{B}_{\mu,0}$ is a subspace of $\mathscr{B}_\mu$. Obviously, $\mathscr{B}_\mu$ is a Banach space under this norm. For $\alpha > 0$, let $\mu(z) = (1-|z|^2)^\alpha$, then $\mathscr{B}_\mu$ is the α-Bloch space $\mathscr{B}^\alpha$. When α=1, it degenerates into the classical Bloch space; if we let $\mu(z) = (1-|z|^2)\ln\frac{e}{1-|z|^2}$, we obtain the logarithmic Bloch space $\mathscr{B}_{1og}$. In [1], [2], [3], [4], [5], more information and research about Bloch spaces can be found.

Many scholars have generalized the Bloch space. For example, Fernández defined the Bloch-Orlicz space using the Yang′s function in [6].

Definition 1.2: Let $\varphi: [0,\infty) \to [0,\infty)$ be an N-function, that is, φ is a strictly increasing convex function with $\varphi(0) = 0$ and $\lim_{t\to\infty}\frac{t}{\varphi(t)} = \lim_{t\to 0}\frac{t}{\varphi(t)} = 0$. For $\alpha > 0$, we define the set $\mathscr{B}_\alpha^\varphi$ as the class of all analytied function f in $\mathbb{D}$ such that:

$$\mathscr{B}_\alpha^\varphi = \left\{ f(z) \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \varphi|\lambda f'(z)| < \infty \right\} \quad (3)$$

some $\lambda > 0$ depended by f .

Obviously, in Definition 1.2, we can observe that $\mathscr{B}_\alpha^\varphi$ get back the $\mathscr{B}^\alpha$ when $\varphi(t) = t$ with $t \geq 0$. More information and research about Bloch-Orlicz spaces can be found in [7], [8], [9], [10]. Since φ is a convex function, it is not difficult to see that the Minkowski′s function:

$$\|f\|_{b_\alpha^\varphi} = \inf\left\{ k > 0 : S_{\varphi,\alpha}\left(\frac{f'}{k}\right) \leq 1 \right\} \quad (4)$$

It is a semi-norm of $\mathscr{B}_\alpha^\varphi$, and in this case, it is called the Luxemburg′s semi-norm, where:

$$S_{\varphi,\alpha}(f) = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \varphi(|f(z)|) \quad (5)$$

In fact, $\mathscr{B}_\alpha^\varphi$ is a Banach space with the norm $\|f\|_{\mathscr{B}_\alpha^\varphi} = |f(0)| + \|f\|_{b_\alpha^\varphi}$.

Definition 1.3: For $\beta \geq 0$， the Bers space $\mathcal{H}_\beta^\infty$ and the little-Bers space $\mathcal{H}_{\beta,0}^\infty$ are defined as follows:

$$\mathcal{H}_\beta^\infty = \left\{ f(z) \in H(\mathbb{D}) : \|f\|_{\mathcal{H}_\beta^\infty} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta f(z) < \infty \right\} \quad (6)$$

$$\mathcal{H}_{\beta,0}^{\infty} = \left\{f(z) \in \mathcal{H}_{\beta}^{\infty}: \lim_{z\to 1^-}(1-|z|^2)^{\beta}|f(z)|\right\} \tag{7}$$

With this norm, $\mathcal{H}_{\beta}^{\infty}$ is a Banach space, and $\mathcal{H}_{\beta,0}^{\infty}$ is a subspace of $\mathcal{H}_{\beta}^{\infty}$. In [11], [12], [13], more information about this space can be found.

Inspired by the definition of the α-Bloch-Orlicz space, this paper defines the β-Bers-Orlicz space $\mathcal{H}_{\beta}^{\infty,\varphi}$:

Definition 1.4: For any $\beta \geq 0$ and $\lambda > 0$ depended by f, if:

$$\mathcal{H}_{\beta}^{\infty,\varphi} = \left\{f(z) \in H(\mathbb{D}): \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\beta}\varphi|\lambda f(z)| < \infty\right\} \tag{8}$$

Then $\mathcal{H}_{\beta}^{\infty,\varphi}$ is called the β-Bers-Orlicz space. Similar to the definition of the β-Bloch-Orlicz space, φ is a convex function and a Minkowski function:

$$\|f\|_{\mathcal{H}_{\beta}^{\infty,\varphi}} = \inf\left\{k > 0: S_{\varphi,\beta}\left(\frac{f}{k}\right) \leq 1\right\} \tag{9}$$

It is a semi-norm of $\mathcal{H}_{\beta}^{\infty,\varphi}$, where the definition of $S_{\varphi,\beta}(f)$ is the same as that in (1).

Let ϕ be an analytic self-mapping function on D. The definition of the composition operator is related to ϕ, denoted as C_{ϕ}. It connects the properties in function theory with those on different spaces. For any $f(z) \in H(\mathbb{D})$ and $z \in \mathbb{D}$, its definition is:

$$C_{\phi}f = f \circ \phi = f(\phi(z)) \tag{10}$$

Definition 1.5: Let $u \in H(\mathbb{D})$ and ϕ be an analytic self - mapping function on $\mathbb{D}$. The definition of the weighted composition operator is related to ϕ and u, denoted as uC_{ϕ}, and it is defined as:

$$uC_{\phi}(f) = u(z)f(\phi(z)) \tag{11}$$

Obviously, in Definition 1.5, uC_{ϕ} get back the C_{ϕ} when $u = 1$. For research results related to uC_{ϕ}, see [14], [15], [16].

In [6], Fernández characterized the boundedness and compactness of composition operators in Bloch-Orlicz-type spaces; In [17], Yang obtained the necessary and sufficient conditions for the boundedness and compactness of integral operators from the Zygmund space to the Bloch-Orlicz space and the Zygmund - Orlicz space. Inspired by scholars such as Fernández and Yang, the aim of this paper is to characterize the boundedness and compactness of $uC_{\phi}(f): \mathscr{B}^{\alpha} \to \mathcal{H}_{\beta}^{\infty,\varphi}$.

In this paper, C represents a positive constant, and its meaning varies in different contexts.

II.PRELIMINARIES

To obtain the main results of this paper, we need the following lemmas and conclusions.

Lemma 2.1: [18] For $0 < \alpha < \infty$, if $f \in \mathscr{B}^{\alpha}$, then for any $z \in \mathbb{D}$, there exists a positive constan $C_1 > 0$ such that:

$$|f(z)| \leq \begin{cases} C_1\|f\|_{\mathscr{B}^{\alpha}}, & 0<\alpha<1; \\ C_1\|f\|_{\mathscr{B}^{\alpha}}\ln\dfrac{2}{1-|z|^2}, & \alpha = 1; \\ \dfrac{C_1\|f\|_{\mathscr{B}^{\alpha}}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1. \end{cases} \tag{12}$$

Using the same proof method as in [19], we can obtain the following lemma:

Lemma 2.2: Let $\varphi: [0,\infty) \to [0,\infty)$ be an N-function, and ϕ be an analytic self-mapping function on $\mathbb{D}$. The operator $uC_{\phi}(f): \mathscr{B}^{\alpha} \to \mathcal{H}_{\beta}^{\infty,\varphi}$ is compact if and only if $uC_{\phi}(f): \mathscr{B}^{\alpha} \to \mathcal{H}_{\beta}^{\infty,\varphi}$ is bounded, and for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathscr{B}^{\alpha}$ that converges uniformly to zero on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\lim_{n\to\infty}\left\|uC_{\phi}(f)\right\|_{\mathcal{H}_{\beta}^{\infty,\varphi}} = 0$.

Lemma 2.3: For any $f \in \mathcal{H}_{\beta}^{\infty,\varphi}\backslash\{0\}$:

$$S_{\varphi,\alpha}\left(\frac{f}{\|f\|_{\mathcal{H}_{\beta}^{\infty,\varphi}}}\right) \leq 1 \tag{12}$$

Proof: If $f \in \mathcal{H}_\beta^{\infty,\varphi}\backslash\{0\}$, then there exists a monotonically decreasing positive sequence $\{\lambda_n\}$ such that for any $\{\lambda_n\}$, we have $\lim_{n\to\infty} \lambda_n = \|f\|_{\mathcal{H}_\beta^{\infty,\varphi}}$. And $S_{\varphi,\beta}\left(\frac{f}{\lambda_n}\right) \leq 1$ for any $n \in \mathbb{N}$. By the definition of $S_{\varphi,\beta}$ and the properties of φ, we know that $S_{\varphi,\beta}\left(\frac{f}{k}\right)$ is monotonically decreasing with respect to k. So:

$$S_n = S_{\varphi,\beta}\left(\frac{f}{\lambda_n}\right) \leq S_{\varphi,\beta}\left(\frac{f}{\|f\|_{\mathcal{H}_\beta^{\infty,\varphi}}}\right) = S \tag{13}$$

Since λ_n is monotonically decreasing, we know that S_n is monotonically increasing and bounded above. Therefore, there exists S′ such that $\lim_{n\to\infty} S_n = S'$. So:

$$S' = \sup\{S_n\} = \sup S_{\varphi,\beta}\left(\frac{f}{\lambda_n}\right) \leq 1 \tag{14}$$

for any $n \in \mathbb{N}$, and:

$$S' \geq S_n \tag{16}$$

Combining (13) and (14) again, we can obtain:

$$S' \leq S \tag{17}$$

By (1), for any $f \in \mathcal{H}_\beta^{\infty,\varphi}$ and z ∈ $\mathbb{D}$, we have:

$$S_n = S_\varphi\left(\frac{f}{\lambda_n}\right) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|f(z)|}{\lambda_n}\right) \leq S' \tag{18}$$

when $n \to \infty$, we have:

$$S = S_{\varphi,\beta}\left(\frac{f}{\|f\|_{\mathcal{H}_\beta^{\infty,\varphi}}}\right) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|f(z)|}{\|f\|_{\mathcal{H}_\beta^{\infty,\varphi}}}\right) \leq S' \tag{19}$$

From (14), (17) and (19), we can obtain:

$$\lim_{n\to\infty} S_n = S' = S = S_{\varphi,\beta}\left(\frac{f}{\|f\|_{\mathcal{H}_\beta^{\infty,\varphi}}}\right) \leq 1 \tag{20}$$

Theorem 2.3 is completely proved.

Corollary 2.4: By Lemma 2.3, for $f \in \mathcal{H}_\beta^{\infty,\varphi}$ and any z ∈ $\mathbb{D}$, we have that:

$$|f(z)| \leq \varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\|f(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} \tag{21}$$

From the definition of the Luxemburg semi-norm and Lemma 2.3, for any $f \in \mathcal{H}_\beta^{\infty,\varphi}$:

$$S_{\varphi,\beta}(f) \leq 1 \Leftrightarrow \|f(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} \leq 1 \tag{22}$$

Corollary 2.5: For any z ∈ $\mathbb{D}$, when:

$$\mu_\beta(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \tag{23}$$

the β-Bers-Orlicz space is isometrically equal to the μ_β-Bers space. In this case, there exists an equivalent norm $\|f(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f(z)|$.

Proof: For any z ∈ $\mathbb{D}$, if $f \in \mathcal{H}_\beta^{\infty,\varphi}\backslash\{0\}$, by Lemma 2.3, we know that：

$$(1-|z|^2)^\beta \varphi\left(\frac{|f(z)|}{\|f\|_{\mathcal{H}_\beta^{\infty,\varphi}}}\right) \leq 1 \tag{24}$$

which means：

$$\mu_\beta(z)|f(z)| \leq \|f(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} \tag{25}$$

By the definition of $\mathcal{H}_\beta^\infty$, we know that $f \in \mathcal{H}_\beta^\infty$, so $\mathcal{H}_\beta^{\infty,\varphi} \in \mathcal{H}_\beta^\infty$, and $\|f(z)\|_{\mathcal{H}_\beta^\infty} \leq \|f(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}}$.

Conversely, if $f \in \mathcal{H}_\beta^\infty$, then by the definition of $\mathcal{H}_\beta^\infty$, we know that：

$$\mu_\beta(z)|f(z)| = \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}|f(z)| < \|f(z)\|_{\mathcal{H}_\beta^\infty} \tag{25}$$

Therefore:

$$S_{\varphi,\beta}\left(\frac{f}{\|f\|_{\mathcal{H}_\beta^\infty}}\right) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|f(z)|}{\|f\|_{\mathcal{H}_\beta^\infty}}\right) \le 1 \tag{26}$$

So $\|f(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} \le \|f(z)\|_{\mathcal{H}_\beta^\infty}$, Let $\lambda = \frac{1}{\|f(z)\|_{\mathcal{H}_\beta^\infty}} > 0$, then Equation (26) becomes:

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta \varphi\left(\frac{|f(z)|}{\lambda}\right) \le \infty \tag{27}$$

By the definition of $\mathcal{H}_\beta^{\infty,\varphi}$, we know that $f \in \mathcal{H}_\beta^{\infty,\varphi}$, that is, $\mathcal{H}_\beta^\infty \in \mathcal{H}_\beta^{\infty,\varphi}$.

Therefore $\|f(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f(z)|$. In conclusion, Corollary (2.5) is proved.

III.MAIN THEOREMS AND THEIR PROOFS

This section studies the boundedness and compactness of weighted composition operators from the α-Bloch space to the β-Bers-Orlicz space. By using the methods of complex analysis and functional analysis, the necessary and sufficient conditions for $uC_\phi(f)\colon \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ to be bounded and compact are obtained. The main conclusions are as follows.

Theorem 3.1: Let $\varphi\colon [0,\infty) \to [0,\infty)$ be an N-function, $u \in H(\mathbb{D})$, and ϕ be an analytic self-mapping function on $\mathbb{D}$. Then $uC_\phi(f)\colon \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is bounded if and only if:

$$K_1 = \sup_{z\in\mathbb{D}} \frac{|u(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \quad for\ 0<\alpha<1 \tag{28}$$

$$K_2 = \sup_{z\in\mathbb{D}} \frac{|u(z)|\ln\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} < \infty, \quad for\ \alpha=1 \tag{29}$$

$$K_3 = \sup_{z\in\mathbb{D}} \frac{|u(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\alpha-1}} < \infty, \quad for\ \alpha>1 \tag{30}$$

Proof: Sufficiency.

(1) For the case $0<\alpha<1$, assuming that (28) is true. for any $f \in \mathscr{B}^\alpha$, combining with the first-term of (12), we can obtain：

$$\begin{aligned} S_{\varphi,\beta}\left(\frac{(uC_\phi f)(z)}{C_1K_1\|f\|_{\mathscr{B}^\alpha}}\right) &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|u(z)||f(\phi(z))|}{C_1K_1\|f\|_{\mathscr{B}^\alpha}}\right) \\ &\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{|u(z)||f(\phi(z))|}{|u(z)|C_1\|f\|_{\mathscr{B}^\alpha}}\right) \\ &\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{|f(\phi(z))|}{C_1\|f\|_{\mathscr{B}^\alpha}}\right) \\ &\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{C_1\|f\|_{\mathscr{B}^\alpha}}{C_1\|f\|_{\mathscr{B}^\alpha}}\right) \\ &\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\right) \le 1 \end{aligned} \tag{28}$$

As known from (22):

$$\left\|\frac{(uC_\phi f)(z)}{C_1K_1\|f\|_{\mathscr{B}^\alpha}}\right\|_{\mathcal{H}_\beta^{\infty,\varphi}} \le 1 \tag{29}$$

Therefore, when $0<\alpha<1$, $\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} \le C_1K_1\|f\|_{\mathscr{B}^\alpha} < \infty$ for any z ∈ $\mathbb{D}$, that is, $uC_\phi(f)\colon \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is bounded.

(2) For $\alpha = 1$, assuming that (29) holds, for any $f \in \mathscr{B}^\alpha$, by combining with the second-term of (12), we know that:

$$S_{\varphi,\beta}\left(\frac{(uC_\phi f)(z)}{C_1K_2\|f\|_{\mathcal{B}^\alpha}}\right) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|u(z)||f(\phi(z))|}{C_1K_2\|f\|_{\mathscr{B}^\alpha}}\right)$$
$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{|u(z)||f(\phi(z))|}{C_1|u(z)|\log\frac{2}{1-|\phi(z)|^2}\|f\|_{\mathscr{B}^\alpha}}\right)$$
$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{C_1\log\frac{2}{1-|\phi(z)|^2}\|f\|_{\mathscr{B}^\alpha}}{C_1\log\frac{2}{1-|\phi(z)|^2}\|f\|_{\mathscr{B}^\alpha}}\right) \tag{30}$$
$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\right) \leq 1$$

Which implies that:

$$\left\|\frac{(uC_\phi f)(z)}{C_1K_2\|f\|_{\mathscr{B}^\alpha}}\right\|_{\mathcal{H}_\beta^{\infty,\varphi}} \leq 1 \tag{31}$$

Similarly, when $\alpha = 1$, $uC_\phi(f)\colon \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is bounded.

(3) For $\alpha > 1$, assuming that (30) holds, for any $f \in \mathscr{B}^\alpha$, by combining with the third-term of (12), we know that:

$$S_{\varphi,\beta}\left(\frac{(uC_\phi f)(z)}{C_1K_3\|f\|_{\mathscr{B}^\alpha}}\right) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|u(z)||f(\phi(z))|}{C_1K_1\|f\|_{\mathscr{B}^\alpha}}\right)$$
$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{(1-|\phi(z)|^2)^{\alpha-1}|u(z)||f(\phi(z))|}{|u(z)|C_1\|f\|_{\mathscr{B}^\alpha}}\right)$$
$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\frac{C_1(1-|\phi(z)|^2)^{\alpha-1}\|f\|_{\mathscr{B}^\alpha}}{C_1(1-|\phi(z)|^2)^{\alpha-1}\|f\|_{\mathscr{B}^{\alpha^\alpha}}}\right) \tag{32}$$
$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)\right) \leq 1$$

Which means that:

$$\left\|\frac{(uC_\phi f)(z)}{C_1K_3\|f\|_{\mathscr{B}^\alpha}}\right\|_{\mathcal{H}_\beta^{\infty,\varphi}} \leq 1 \tag{33}$$

Similarly, when $\alpha > 1$, $uC_\phi(f)\colon \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is bounded.

Necessity.

Suppose that $uC_\phi(f)\colon \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is bounded. Then there exists $C > 0$ such that for any $f \in \mathscr{B}^\alpha$:

$$\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} \leq C\|f\|_{\mathscr{B}^\alpha} \tag{34}$$

That is to say:

$$\left\|\frac{(uC_\phi f)(z)}{C\|f\|_{\mathscr{B}^\alpha}}\right\|_{\mathcal{H}_\beta^{\infty,\varphi}} \leq 1 \tag{35}$$

In light of (22):

$$S_{\varphi,\beta}\left(\frac{(uC_\phi f)(z)}{C\|f\|_{\mathscr{B}^\alpha}}\right) = S_{\varphi,\beta}\left(\frac{u(z)f(\phi(z))}{C\|f\|_{\mathscr{B}^\alpha}}\right) \leq 1 \tag{36}$$

(1) For $0 < \alpha < 1$, set the test function $f_0(z) = 1 \in \mathscr{B}^\alpha$, and substituting it into (3.4) gives:

$$S_{\varphi,\beta}\left(\frac{u(z)f_0(\phi(z))}{C\|f_0\|_{\mathscr{B}^\alpha}}\right) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|u(z)||f_0(\phi(z))|}{C\|f_0\|_{\mathscr{B}^\alpha}}\right)$$
$$= \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|u(z)|}{C}\right) \tag{37}$$
$$\leq 1$$

Then:

$$M_1 = \sup_{z\in\mathbb{D}} \frac{|u(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} \le C < \infty \tag{38}$$

Therefore, when $0 < \alpha < 1$, (28) holds.

(2) For $\alpha = 1$, Letting $a \in \mathbb{D}$, define the function:

$$f_a(z) = \ln \frac{2}{1-\overline{\phi(a)}z}, z \in \mathbb{D} \tag{39}$$

Then:

$$(1-|z|^2)\left|f_a^{'}(z)\right| = (1-|z|^2)\frac{|\phi(a)|}{\left|1-\overline{\phi(a)}z\right|} \le 2 < \infty \tag{40}$$

This implies that $f_a(z) \in \mathscr{B}^\alpha$ and $\|f_a(z)\|_{\mathscr{B}^\alpha} \le 2 + \ln 2$. Substituting $f_a(z)$ into (36) gives:

$$S_{\varphi,\beta}\left(\frac{u(z)f_a(\phi(z))}{C\|f_a\|_{\mathscr{B}^\alpha}}\right) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\varphi\left(\frac{|u(z)|\left|f_a(\phi(z))\right|}{C\|f_a\|_{\mathscr{B}^\alpha}}\right) \le 1 \tag{41}$$

Then：

$$(1\text{-}|a|^2)^\beta\varphi\left(\frac{|u(a)|f_a(\phi(a))}{C\|f_a\|_{\mathscr{B}^\alpha}}\right) = (1\text{-}|a|^2)^\beta\varphi\left(\frac{|u(a)|\ln\frac{2}{1-|\phi(z)|^2}}{C\|f_a\|_{\mathscr{B}^\alpha}}\right) \le 1 \tag{42}$$

So:

$$\frac{|u(a)|\ln\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right)} \le C\|f_a\|_{\mathscr{B}^\alpha} < \infty \tag{43}$$

Since $a \in \mathbb{D}$ is arbitrary, thus we say that:

$$\sup_{a\in\mathbb{D}} \frac{|u(a)|\ln\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right)} < \infty \tag{44}$$

Therefore, when $\alpha = 1$, (29) holds.

(3) For the case $\alpha > 1$, Letting $a \in \mathbb{D}$, define the function:

$$f_a(z) = \frac{1}{\left(1-\overline{\phi(a)}z\right)^{\alpha-1}}, z \in \mathbb{D} \tag{45}$$

Indeed, it holds：

$$(1-|z|^2)^\alpha\left|f_a^{'}(z)\right| = (1-|z|^2)^\alpha\left|\frac{(\alpha-1)\overline{\phi(a)}}{\left(1-\overline{\phi(a)}z\right)^\alpha}\right| \le |\alpha-1|2^\alpha < \infty \tag{46}$$

This implies that $f_a(z) \in \mathscr{B}^\alpha$ and $\|f_a(z)\|_{\mathscr{B}^\alpha} \le |\alpha-1|2^\alpha + 1$, Similarly, substituting $f_a(z)$ into (36) gives:

$$\sup_{a\in\mathbb{D}} \frac{|u(a)|}{\varphi^{-1}\left(\frac{1}{(1-|a|^2)^\beta}\right)(1-|\phi(a)|^2)^{\alpha-1}} < \infty \tag{47}$$

Therefore, when $\alpha > 1$, (30) holds.

In conclusion, Theorem 3.1 is fully proved.

Theorem 3.2: Let $\varphi: [0,\infty) \to [0,\infty)$ be an N-function, $u \in H(\mathbb{D})$, and ϕ is an analytic self-mapping function on $\mathbb{D}$. Then $uC_\phi(f): \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is compact if and only if: $uC_\phi(f): \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is bounded and:

$$\lim_{\phi(z)\to 1} \frac{|u(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)} = 0, \quad for\ 0 < \alpha < 1 \tag{48}$$

$$\lim_{\phi(z)\to 1}\frac{|u(z)|\ln\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}=0,\quad for\ \alpha=1 \tag{49}$$

$$\lim_{\phi(z)\to 1}\frac{|u(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\alpha-1}}=0,\quad for\ \alpha>1 \tag{50}$$

Proof: Sufficiency.

Suppose that $uC_\phi(f)\colon \mathscr{B}^\alpha\to\mathcal{H}_\beta^{\infty,\varphi}$ is bounded, and (48), (49), (40) hold. Let $\{f_n\}$ be a bounded sequence in $\mathscr{B}^\alpha$ that converges uniformly to zero on the compact subsets of $\mathbb{D}$. Then there exists $K>0$ such that $\sup_{n\in\mathbb{N}}\|f_n\|_{\mathscr{B}^\alpha}\le K$. By Lemma 2.2, to prove that $uC_\phi(f)\colon \mathscr{B}^\alpha\to\mathcal{H}_\beta^{\infty,\varphi}$ is compact, it suffices to show that $\lim_{n\to\infty}\|(uC_\phi)(f)\|_{\mathcal{H}_\beta^{\infty,\varphi}}=0$. For $0<r<1$, obviously:

$$\begin{aligned}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}}&=\sup_{z\in\mathbb{D}}\frac{1}{\varphi^{-1}((1-|z|^2)^\beta)}\left|(uC_\phi f_n)(z)\right|\\&=\sup_{z\in\mathbb{D}}\frac{1}{\varphi^{-1}((1-|z|^2)^\beta)}\left(|u(z)|\left|f_n(\phi(z))\right|\right)\\&=\sup_{\{z\in\mathbb{D}:|\phi(z)|\le r\}}\frac{1}{\varphi^{-1}((1-|z|^2)^\beta)}\left(|u(z)|\left|f_n(\phi(z))\right|\right)\\&+\sup_{\{z\in\mathbb{D}:|\phi(z)|>r\}}\frac{1}{\varphi^{-1}((1-|z|^2)^\beta)}\left(|u(z)|\left|f_n(\phi(z))\right|\right)\end{aligned} \tag{51}$$

（1） By Theorem 3.1 and the boundedness of $uC_\phi(f)\colon \mathscr{B}^\alpha\to\mathcal{H}_\beta^{\infty,\varphi}$, we prove that (28) holds when $0<\alpha<1$, that is $K_1<\infty$. From the fact that (48) holds, for any $\varepsilon>0$, there exists $0<r_1<1$ such that:

$$\frac{|u(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}<\varepsilon, whenever\ \ \phi(z)>r_1 \tag{52}$$

Due to Lemma 2.1, when $0<\alpha<1$, $\left|f_n(\phi(z))\right|\le C_1\left\|f_n(\phi(z))\right\|_{\mathscr{B}^\alpha}$, so:

$$\begin{aligned}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}}&=\sup_{\{z\in\mathbb{D}:|\phi(z)|\le r_1\}}K_1\left|f_n(\phi(z))\right|+\sup_{\{z\in\mathbb{D}:|\phi(z)|>r_1\}}C_1K\frac{|u(z)|}{\varphi^{-1}((1-|z|^2)^\beta)}\\&\le\sup_{\{z\in\mathbb{D}:|\phi(z)|\le r_1\}}K_1\left|f_n(\phi(z))\right|+C_1K\varepsilon\end{aligned} \tag{53}$$

Since $\{f_n\}$ converges uniformly to zero on the compact subsets of $\mathbb{D}$, that is, $\lim_{n\to\infty}\sup_{\{z\in\mathbb{D}:|\phi(z)|\le r_1\}}f_n(\phi(z))=0$. So, by the arbitrariness of ε, we conclude that:

$$\lim_{n\to\infty}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}}=0 \tag{54}$$

By Lemma 2.2, when $0<\alpha<1$, $uC_\phi(f)\colon \mathscr{B}^\alpha\to\mathcal{H}_\beta^{\infty,\varphi}$ is compact.

（2） By Theorem 3.1 and the boundedness of $uC_\phi(f)\colon \mathscr{B}^\alpha\to\mathcal{H}_\beta^{\infty,\varphi}$, we prove that (29) holds when $\alpha=1$, that is $K_2<\infty$. From the fact that (49) holds, for any $\varepsilon>0$, there exists $0<r_2<1$ such that:

$$\frac{|u(z)|\ln\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}<\varepsilon, whenever\ \ \phi(z)>r_2 \tag{55}$$

For the case $\alpha=1$, by Lemma 2.1, $\left|f_n(\phi(z))\right|\le C_1\left\|f_n(\phi(z))\right\|_{\mathscr{B}^\alpha}\ln\frac{2}{1-|\phi(z)|^2}$, so:

$$\begin{aligned}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}}&=\sup_{\{z\in\mathbb{D}:|\phi(z)|\le r_2\}}K_2\left|f_n(\phi(z))\right|+\sup_{\{z\in\mathbb{D}:|\phi(z)|>r_2\}}C_1K\frac{|u(z)|\ln\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}((1-|z|^2)^\beta)}\\&\le\sup_{\{z\in\mathbb{D}:|\phi(z)|\le r_2\}}K_2\left|f_n(\phi(z))\right|+C_1K\varepsilon\end{aligned} \tag{56}$$

Similarly:

$$\lim_{n\to\infty}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}}=0 \tag{57}$$

By Lemma 2.2, when $\alpha = 1$, $uC_\phi(f): \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is compact.

(3) By Theorem 3.1 and the boundedness of $uC_\phi(f): \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$, we prove that (30) holds when $\alpha = 1$, that is $K_3 < \infty$. From the fact that (50) holds, for any $\varepsilon > 0$, there exists $0 < r_3 < 1$ such that:

$$\frac{|u(z)|}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)(1-|\phi(z)|^2)^{\alpha-1}} < \varepsilon, whenever\ \ \phi(z) > r_3 \tag{58}$$

For the case $\alpha > 1$, by Lemma 2.1, $|f_n(\phi(z))| \le \frac{C_1\|f_n(\phi(z))\|_{\mathscr{B}^\alpha}}{(1-|\phi(z)|^2)^{\alpha-1}}$, so:

$$\begin{aligned}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} &= \sup_{\{z\in\mathbb{D}:|\phi(z)|\le r_3\}} K_3|f_n(\phi(z))| + \sup_{\{z\in\mathbb{D}:|\phi(z)|> r_3\}} C_1 K\frac{|u(z)|}{(1-|\phi(z)|^2)^{\alpha-1}\varphi^{-1}((1-|z|^2)^\beta)} \\ &\le \sup_{\{z\in\mathbb{D}:|\phi(z)|\le r_3\}} K_3|f_n(\phi(z))| + C_1K\varepsilon\end{aligned} \tag{59}$$

Similarly:

$$\lim_{n\to\infty}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} = 0 \tag{60}$$

By Lemma 2.2, when $\alpha > 1$, $uC_\phi(f): \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is compact.

Necessity.

Suppose that $uC_\phi(f): \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is compact. Obviously, $uC_\phi(f): \mathscr{B}^\alpha \to \mathcal{H}_\beta^{\infty,\varphi}$ is bounded. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k\to\infty}|\phi(z_k)| = 1$. According to the definition of the Bers-Orlicz space, we have:

$$\begin{aligned}\|(uC_\phi f)(z)\|_{\mathcal{H}_\beta^{\infty,\varphi}} &= \sup_{z\in\mathbb{D}}\frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}|u(z)f(\phi(z))| \\ &\ge \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^\beta}\right)}|u(z_k)f(\phi(z_k))|\end{aligned} \tag{61}$$

(1) When $0 < \alpha < 1$, define the function:

$$f_{1,k}(z) = \frac{1-|\phi(z_k)|^2}{1-\overline{\phi(z_k)}z} \tag{62}$$

Then:

$$(1-|z|^2)^\alpha|f_{1,k}'(z)| = (1-|z|^2)^\alpha\left|\frac{\overline{\phi(z_k)}(1-|\phi(z_k)|^2)}{(1-\phi(z_k)z)^2}\right| < 4^\alpha < \infty \tag{63}$$

Therefore, $f_{1,k}(z) \in \mathscr{B}^\alpha$. Moreover, for any $k \in \mathbb{N}$, on the compact subsets of $\mathbb{D}$, we have:

$$\lim_{k\to\infty} f_{1,k}(z) = \lim_{k\to\infty}\frac{1-|\phi(z_k)|^2}{1-\overline{\phi(z_k)}z} = 0 \tag{64}$$

That is, $f_{1,k}(z)$ converges uniformly to 0 on the compact subsets of $\mathbb{D}$. By Lemma 2.2, $\lim_{n\to\infty}\|(uC_\phi)(f_{1,k})\|_{\mathcal{H}_\beta^{\infty,\varphi}} = 0$. Substitute $f_{1,k}(z)$ into (61):

$$\begin{aligned}\|(uC_\phi f_{1,k})(z_k)\|_{\mathcal{H}_\alpha^{\infty,\varphi}} &\ge \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^\alpha}\right)}|u(z_k)|f_{1,k}(\phi(z_k)) \\ &\ge \frac{|u(z_k)|}{\varphi^{-1}\left(\frac{1}{(1-|z_k|^2)^\alpha}\right)}\end{aligned} \tag{65}$$

So, when $0 < \alpha < 1$, (47) holds.

(2) When $\alpha = 1$, define the function:

$$f_{2,k}(z) = \left(\ln\frac{2}{1-\overline{\phi(z_k)}z}\right)^2\left(\ln\frac{2}{1-|\phi(z_k)|^2}\right)^{-1} \tag{66}$$

According to [17] for any $a \in \mathbb{D}$:

$$\left|\frac{2\bar{a}}{1-\bar{a}z}\left(\ln\frac{2}{1-\bar{a}z}\right)\left(\ln\frac{2}{1-|a|^2}\right)^{-1}\right| \le \frac{2}{1-|z|} \tag{67}$$

So:

$$\begin{aligned}(1-|z|^2)\left|f'_{2,k}(z)\right| &= (1-|z|^2)\left|\frac{2\overline{\phi(z_k)}}{1-\overline{\phi(z_k)}z}\right|\left|\left(\ln\frac{2}{1-\overline{\phi(z_k)}z}\right)\left(\ln\frac{2}{1-|\phi(z_k)|^2}\right)^{-1}\right| \\ &\le \frac{2(1-|z|^2)}{1-|z|} < 4 < \infty\end{aligned} \tag{68}$$

This implies that $f_{2,k}(z) \in \mathscr{B}^{\alpha}$. For any $k \in \mathbb{N}$ and $|z| < r < 1$, as $k \to \infty$, we have:

$$\left|f_{2,k}(z)\right| = \frac{\left|\ln\frac{2}{1-\overline{\phi(z_k)}z}\right|^2}{\ln\frac{2}{1-|\phi(z_k)|^2}} \le \frac{\left(\ln\frac{2}{1-r}+C\right)^2}{\ln\frac{2}{1-|\phi(z_k)|^2}} \to 0 \quad (as\ k \to \infty) \tag{69}$$

That is, $f_{2,k}(z)$ converges uniformly to 0 on the compact subsets of $\mathbb{D}$. By Lemma 2.2, $\lim_{n\to\infty}\left\|(uC_{\phi})(f_{2,k})\right\|_{\mathcal{H}_{\beta}^{\infty,\varphi}} = 0$. Substitute $f_{2,k}(z)$ into (61), Similarly, when $\alpha = 1$, (48) holds.

（3） When $\alpha = 1$, define the function:

$$f_{3,k}(z) = \frac{1-|\phi(z_k)|^2}{\left(1-\overline{\phi(z_k)}z\right)^{\alpha}} \tag{70}$$

Then:

$$\begin{aligned}(1-|z|^2)^{\alpha}\left|f'_{3,k}(z)\right| &= (1-|z|^2)^{\alpha}\left|\frac{\alpha\overline{\phi(z_k)}(1-|\phi(z_k)|^2)}{\left(1-\overline{\phi(z_k)}z\right)^{\alpha+1}}\right| \\ &\le \alpha\frac{(1-|z|)^{\alpha}(1+|z|)^{\alpha}(1-|\phi(z_k)|)(1+|\phi(z_k)|)}{(1-|\phi(z_k)|)(1-|z|)^{\alpha}} \\ &\le \alpha 2^{\alpha+1} < \infty\end{aligned} \tag{71}$$

Therefore, $f_{3,k}(z) \in \mathscr{B}^{\alpha}$. Moreover, for any $k \in \mathbb{N}$, on the compact subsets $\mathbb{D}$, we have:

$$\lim_{k\to\infty}\frac{1-|\phi(z_k)|^2}{\left(1-\overline{\phi(z_k)}z\right)^{\alpha}} = 0 \tag{72}$$

That is, $f_{3,k}(z)$ converges uniformly to 0 on the compact subsets of $\mathbb{D}$. By Lemma 2.2, $\lim_{n\to\infty}\left\|(uC_{\phi})(f_{3,k})\right\|_{\mathcal{H}_{\beta}^{\infty,\varphi}} = 0$. Substitute $f_{3,k}(z)$ into (61), Similarly, when $\alpha > 1$, (49) holds.

In conclusion, Theorem 3.2 is fully proved.

IV.CONCLUSION

Compactness and boundedness are very important properties for function spaces. In this paper, the Bers-Orlicz space is defined. The exponent α is divided into three non-overlapping parts. By using the methods of complex analysis and functional analysis, the necessary and sufficient conditions for the boundedness and compactness of $uC_{\phi}(f)$: $\mathscr{B}^{\alpha} \to \mathcal{H}_{\beta}^{\infty,\varphi}$ in different parts are obtained. However, this paper only studies the general weighted composition operator, with a relatively narrow scope. Further research in this paper could focus on the generalized weighted composition operator, aiming to find out under which conditions the generalized weighted composition operator is bounded and compact in the Bers-Orlicz space.

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