On \((LCS)_n\)-manifold admitting \(M\)-projective curvature tensor

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Abstract: The objective of this paper is to study \(M\)-projective flat, \(M\)-projective semisymmetric, \(M\)-projective Ricci pseudosymmetric \((LCS)_n\)-manifold. Further we study \((LCS)_n\)-manifold satisfying \(W^* \cdot R = 0\) and \(W^* \cdot S = 0\).

Key Words: \((LCS)_n\)-manifold, \(M\)-projective curvature tensor, \(M\)-projective flat, \(M\)-projective semisymmetric, \(M\)-projective Ricci pseudosymmetric.

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I. Introduction

In 2003, Shaikh [15] introduced and studied Lorentzian concircular structure manifolds (briefly \((LCS)_n\)-manifolds) with an example, which generalizes the notion of \(LP\)-Sasakian manifolds introduced by Matsumoto [8]. Also Shaikh et al. ([16, 17, 18, 19]), Prakash [13], Yadav [26] studied various types of \((LCS)_n\)-manifolds by imposing curvature restrictions.

In 1926, the concept of local symmetry of a Riemannian manifold was started by Cartan [1]. This notion has been used in several directions by many authors such as recurrent manifolds by Walker [25] and semi-symmetric manifold by Szabo [21], pseudosymmetric manifold by Chaki [2], pseudosymmetric spaces by Deszcz [6], weakly symmetric manifold by Tamassy and Binh [23], weakly symmetric Riemannian spaces by Selberg [14]. Despite, the notions of pseudosymmetric and weak symmetry respectively by Chaki and Deszcz and Selberge and Tamassy and Binh are different. As a mild version of local symmetry, Takahashi [22] introduced the notion of \(\phi\)-symmetry on a Sasakian manifold. In 2003, De et al. [5] introduced the concept of \(\phi\)-recurrent Sasakian manifold, which generalizes the notion of \(\phi\)-symmetry.

In 1971, Pokhariyal and Mishra [12] defined a tensor field \(W^*\) on a Riemannian manifold given by

\(W^*(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]\)

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Such a tensor field \(W^*\) is known as \(M\)-projective curvature tensor. Ojha [10, 11] studied \(M\)-projective curvature tensor on Sasakian and Kaehler manifold. The properties of \(M\)-projective curvature tensor were also studied on different manifolds by Chaubey [3, 4], Venkatesha [24] and others.

Motivated by the above studies, we made an attempt to study \(M\)-projective curvature tensor on \((LCS)_n\)-manifold.

The present paper is organized as follows: Section 2 is equipped with some preliminaries of \((LCS)_n\)-manifold. In section 3, we have proved that if an \(n\)-dimensional \((LCS)_n\)-manifold \(M^n (n > 1)\) is \(M\)-projective flat if and only if the manifold is of constant scalar curvature \((\alpha^2 - \rho) \neq 0\). Section 4 deals with the study of \((LCS)_n\)-manifold satisfying \(W^* \cdot R = 0\). We study \(M\)-projective semisymmetric \((LCS)_n\)-manifold in section 5. Section 6 is devoted to the study of \(M\)-projective Ricci-pseudosymmetric \((LCS)_n\)-manifold. In the last section, we study \((LCS)_n\)-manifold satisfying \(W^* \cdot S = 0\).

II. Preliminaries

An \(n\)-dimensional Lorentzian manifold \(M^n\) is a smooth connected para-compact Hausdorff manifold with a Lorentzian metric \(g\) of type \((0,2)\) such that for each point \(p \in M\), the tensor \(g_p : T_p(M^n) \times T_p(M^n) \rightarrow R\) is a non-degenerate inner product of signature \((-+,+,\ldots,+)\), where \(T_p(M^n)\) denotes the tangent space of \(M^n\) at \(p\) and \(R\) is the real number space [15, 9]. In a Lorentzian manifold \((M^n, g)\), a vector field \(P\) defined by
for any vector field $X \in \chi(M^n), (\chi(M^n))$, being the Lie algebra of vector fields on $M^n$ is said to be a concircular vector field [20] if

$$\nabla_X A(Y) = \alpha[g(X, Y) + \omega(X)A(Y)],$$

where $\alpha$ is a non-zero scalar function, $A$ is a 1-form and $\omega$ is a closed 1-form.

Let $M^n$ be a Lorentzian manifold admitting a unit time like concircular vector field $\xi$, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1.$$

Since $\xi$ is a unit concircular vector field, there exists a non-zero 1-form $\eta$ such that for

$$g(X, \xi) = \eta(X),$$

the equation of the following form holds

$$\nabla_X \eta(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \quad (\alpha \neq 0)$$

for all vector fields $X$ and $Y$. Here $\nabla$ denotes the covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha$ is a non-zero scalar function satisfying

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho \eta(X),$$

$\rho$ being a certain scalar function given by

$$\rho = -\alpha \eta.$$

If we put

$$\phi X = \frac{1}{\rho} \nabla_X \xi,$$

then from (2.3) and (2.5) we have

$$\phi^2 X = X + \eta(X)\xi,$$

$$\eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

from which it follows that $\phi$ is a symmetric $(1,1)$-tensor, called the structure tensor of the manifold. Thus the Lorentzian manifold $M$ together with the unit timelike concircular vector field $\xi$, its associated 1-form $\eta$ and $(1,1)$-tensor field $\phi$ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [15].

Especially, if we take $\alpha = 1$, then we obtain the LP-Sasakian structure of Matsumoto [8].

In a $(LCS)_n$-manifold, the following relations hold [15]:

$$\eta(R(X, Y)Z) = (a^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$

$$R(X, Y)\xi = (a^2 - \rho)[\eta(Y)X - \eta(X)Y],$$

$$R(X, \xi)Z = (a^2 - \rho)[\eta(Z)X - g(X, Z)\xi],$$

$$R(\xi, X)Y = (a^2 - \rho)[g(X, Y)\xi - \eta(Y)X],$$

$$R(\xi, X)\xi = (a^2 - \rho)[X + \eta(X)\xi],$$

$$S(X, \xi) = (n - 1)(a^2 - \rho)\eta(X), Q\xi = (n - 1)(a^2 - \rho)\xi,$$

$$\nabla_X \phi(Y) = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X],$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)(a^2 - \rho)\eta(X)\eta(Y),$$
for all vector fields $X, Y, Z$ and $R, S$ respectively denotes the curvature tensor and the Ricci
tensor of the manifold.

A $(LCS)_n$ manifold $M^n$ is said to be a $\eta$-Einstein manifold if the following condition

$$(2.16) \quad S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y),$$

holds on $M^n$. Here $\alpha$ and $\beta$ are smooth functions. If $\beta = 0$ then the manifold reduces to an Einstein manifold.

From (1.1), we have

$$(2.17) \quad \eta(W^*(\xi,Y)Z) = \frac{1}{2(n-1)} S(Y,Z) - \frac{1}{2} g(Y,Z),$$

$$(2.18) \quad \eta(W^*(X,\xi)Z) = -\frac{1}{2(n-1)} S(X,Z) - \frac{3}{2} (\alpha^2 - \rho) g(X,Z),$$

$$(2.19) \quad \eta(W^*(X,Y)\xi) = 0,$$

$$(2.20) \quad W^*(X,Y)\xi = 0, \quad W^*(X,\xi)\xi = 0, \quad W^*(\xi,\xi)Z = 0,$$

$$(2.21) \quad W^*(X,\xi)Z = -\frac{1}{2(n-1)} S(X,Z) - \frac{1}{2} (\alpha^2 - \rho) g(X,Z),$$

$$(2.22) \quad W^*(X,\xi)Z = \frac{1}{2(n-1)} S(X,Z)\xi - \frac{1}{2} (\alpha^2 - \rho) g(X,Z)\xi,$$

$$(2.23) \quad W^*(X,\xi)Z = (n-1)\alpha (\alpha^2 - \rho) [g(U,X) + \eta(U)\eta(X)] - \alpha S(X,\phi U).$$

III. $(LCS)_n$-manifold satisfying $W^* = 0$

**Definition 3.1:** An $n$-dimensional, $(n > 3)$ $(LCS)_n$-manifold $M^n$ is said to be $M$-projective flat if the $M$-projective curvature tensor $W^* = 0$.

**Theorem 3.1:** An $n$-dimensional $(LCS)_n$-manifold $M^n$ is $M$-projective flat if and only if the manifold is of constant scalar curvature $(\alpha^2 - \rho) \neq 0$.

**Proof:** Suppose $(LCS)_n$-manifold is $M$-projective flat. Then from (1.1), we get

$$(3.1) \quad R(X,Y)Z = \frac{1}{2(n-1)} [S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY].$$

By replacing $Z = \xi$ in (3.1) and making use of (2.7), (2.9) and (2.13), we obtain

$$(3.2) \quad \eta(Y)QX - \eta(X)QY = (n-1)(\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y].$$

Again treating $Y = \xi$ in (3.2) and using (2.7) and (2.13), we get

$$(3.3) \quad QX = (n-1)(\alpha^2 - \rho)X,$$

which implies

$$(3.4) \quad S(X,Y) = (n-1)(\alpha^2 - \rho)g(X,Y).$$

In view of (3.3) and (3.4), (3.1) gives

$$(3.5) \quad R(X,Y)Z = (\alpha^2 - \rho)[g(Y,Z)X - g(X,Z)Y].$$
Conversely, from the above equations (3.3), (3.4) and (3.5), (1.1) gives

\[ W^*(X,Y)Z = 0, \]

provided \((\alpha^2 - \rho) \neq 0.\)

This completes the proof of the theorem.

**IV. \((\text{LCS})_n\)-manifold satisfying \(W^* \cdot R = 0\)**

Let \(M^n\) be an \(n\)-dimensional \((n > 3)\) \((\text{LCS})_n\)-manifold satisfying \(W^* \cdot R = 0.\) Then we have

\[ (W^*(\xi, Y) \cdot R)(U, V)T = W^*(\xi, Y)R(U, V)T - R(W^*(\xi, Y)U, V)T \]

\[ - R(U, W^*(\xi, Y)V)T - R(U, V)W^*(\xi, Y)T = 0. \]

**Theorem 4.1:** If \(n\)-dimensional \((\text{LCS})_n\)-manifold satisfies \((W^*(\xi, X) \cdot R) = 0,\) then the manifold is Einstein manifold with scalar curvature \(r = n(n - 1)(\alpha^2 - \rho),\) provided \((\alpha^2 - \rho) \neq 0.\)

**Proof:** By taking \(T = \xi\) and using (2.9), the expression (4.1) reduces to

\[ (\alpha^2 - \rho)[\eta(W^*(\xi, Y)V - \eta(W^*(\xi, Y)V)U] = 0. \]

As \((\alpha^2 - \rho) \neq 0,\) taking inner product of (4.2) with \(\xi\) and then using (2.17), we obtain

\[ (\alpha^2 - \rho)[\frac{1}{2(n-1)}S(Y, U)\eta(V) - \frac{1}{2(n-1)}S(Y, V)\eta(U) + \frac{1}{2}(\alpha^2 - \rho)g(Y, V)\eta(U) \]

\[ - \frac{1}{2}(\alpha^2 - \rho)g(Y, U)\eta(V)] = 0. \]

Replacing \(V\) by \(\xi\) in (4.3), we have

\[ S(Y, U) = (n - 1)(\alpha^2 - \rho)g(Y, U). \]

Contracting (4.4), we get

\[ r = n(n - 1)(\alpha^2 - \rho). \]

Hence, the theorem.

**V. \(M\)-projective semisymmetric \((\text{LCS})_n\)-manifold**

An \((\text{LCS})_n\)-manifold is said to be \(M\)-projective symmetric if \(\nabla W^* = 0,\) and it is called \(M\)-projective semisymmetric if

\[ (R(X, Y) \cdot W^*) (U, V)G = 0. \]

**Theorem 5.1:** If in an \(n\)-dimensional \((\text{LCS})_n\)-manifold, the relation \(R \cdot W^* = 0\) holds with the condition \((\alpha^2 - \rho) \neq 0,\) then the manifold is Einstein manifold and the scalar curvature \(r\) of such a manifold is given by

\[ r = (\alpha^2 - \rho)\frac{n(n-1)(n+4)}{(n+1)}, \]

providing \((\alpha^2 - \rho) \neq 0.\)

**Proof:** Let \(M^n\) be a \(M\)-projective semisymmetric. Then from (5.1), we have


\[ - W^*(U, V)R(\xi, Y)G = 0. \]

In view of (2.11) the above expression reduces to,

\[ (\alpha^2 - \rho)[g(Y, W^*(U, V)G)\xi - \eta(W^*(U, V)G)Y - g(Y, U)W^*(\xi, V)G \]

\[ - g(Y, G)W^*(\xi, V)G = 0. \]
Taking inner product of the above equation with $\xi$ and then using (2.2) and (2.7), we get

\begin{equation}
(\alpha^2 - \rho)[-g(Y,V)W^*(U,V)G - \eta(W^*(U,V)G)\eta(Y) - g(Y,U)\eta(W^*(\xi,V)G) + \eta(U)\eta(W^*(Y,V)G) - g(Y,G)\eta(W^*(U,V)\xi) + \eta(G)\eta(W^*(U,V)Y)] = 0,
\end{equation}

which on using (2.8), (2.17) - (2.19), gives

\begin{equation}
(\alpha^2 - \rho)[-R(U,V,G,Y) + \frac{1}{2(n-1)}S(V,G)g(U,Y) - S(U,G)\eta(V)\eta(U) - S(U,G)\eta(\eta(U)\eta(V)\eta(V) - g(U,G)\eta(\eta(U)\eta(V) + S(U,G)\eta(U)\eta(V)) - (n-1)(\alpha^2 - \rho)\eta(Y)\eta(U) + (n-1)(\alpha^2 - \rho)g(Y,G)\eta(V)\eta(U) + S(U,G)g(Y,V) - S(G,U)\eta(Y)\eta(V) + S(U,Y)\eta(G)\eta(V) - S(V,Y)\eta(U)\eta(G) - (n-1)(\alpha^2 - \rho)\eta(G)\eta(V) - (n-1)(\alpha^2 - \rho)g(Y,U)\eta(G)\eta(V) + (n-1)(\alpha^2 - \rho)g(U,G)\eta(Y)\eta(V) + \frac{1}{2}(\alpha^2 - \rho)g(Y,Y)\eta(G)\eta(Y) + 2(\alpha^2 - \rho)g(U,G)\eta(Y)\eta(V) - 2(\alpha^2 - \rho)g(U,Y)\eta(G)\eta(V) + \frac{1}{2}(\alpha^2 - \rho)g(Y,U)\eta(G)\eta(Y) + \frac{3}{2}(\alpha^2 - \rho)g(U,G)g(Y,V) + (\alpha^2 - \rho)g(Y,V)\eta(U)\eta(G)\eta(U)\eta(G)] = 0.
\end{equation}

Contracting (5.5), we obtain

\begin{equation}
S(U,Y) = (\alpha^2 - \rho)\frac{(n-1)(n+1)}{(n+1)}g(U,Y).
\end{equation}

Hence the theorem.

VI. \textbf{M-projective Ricci pseudosymmetric (LCS)\textsubscript{n}-manifold}

\textbf{Definition 6.1:} An $n$-dimensional Riemannian manifold $(M^n, g)$ is said to be Ricci pseudosymmetric \cite{7} if the condition

\begin{equation}
R(U,V) \cdot S(Z,T) = L_S[(((U \wedge V) \cdot S)(Z,T)],
\end{equation}

holds on $U_S = \{x \in M : S \neq \frac{r}{n}g \text{ at } x\}$, where $L_S$ is some function on $U_S$.

\textbf{Definition 6.2:} An $n$-dimensional (LCS)\textsubscript{n}-manifold $(M^n, g)$ is said to be $M$-projectively Ricci pseudosymmetric if $M$-projective curvature tensor $W^*$ satisfies

\begin{equation}
(W^*(U,V) \cdot S)(Z,T) = L_S[(((U \wedge V) \cdot S)(Z,T)],
\end{equation}

Or

\begin{equation}
\end{equation}

\textbf{Theorem 6.1:} If an (LCS)\textsubscript{n}-manifold $M^n$ is $M$-projective Ricci pseudosymmetric with restrictions $V = T = \xi$, then either $L_S = \frac{(\alpha^2 - \rho)}{2}$ or the manifold is Einstein manifold, provided $(\alpha^2 - \rho) \neq 0$. 

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Proof: Restricting $V = T = \xi$ in (6.2) and making use of (1.1), (2.7), (2.10), (2.13), we obtain

\begin{equation}
[L_S - \frac{(\alpha^2 - \rho)}{2}](S(U, Z) + (n - 1)(\alpha^2 - \rho)g(U, Z)) = 0,
\end{equation}

which implies either $L_S = \frac{(\alpha^2 - \rho)}{2}$ or

\begin{equation}
S(U, Z) = (n - 1)(\alpha^2 - \rho)g(U, Z).
\end{equation}

Thus, we proved the theorem.

VII. $(LCS)_n$-manifold satisfying the condition $W^* \cdot S = 0$

Let us suppose that $M^n$ satisfies $W^* \cdot S = 0$. Then

\begin{equation}
S(W^*(X, Y)Z, U) + S(Z, W^*(X, Y)U) = 0.
\end{equation}

**Theorem 7.1:** If $(LCS)_n$-manifold satisfies the condition $W^* \cdot S = 0$, then it is Einstein manifold, provided $(\alpha^2 - \rho) \neq 0$.

**Proof:** Putting $X = \xi$ and in view of (1.1), the equation (7.1) gives

\begin{equation}
(\alpha^2 - \rho)(-\frac{1}{2}S(Y, Z)\eta(U) - \frac{1}{2}S(Y, U)\eta(Z) + \frac{1}{2}(n - 1)(\alpha^2 - \rho)g(Y, Z)\eta(U)
\end{equation}

\begin{equation}
+ \frac{1}{2}(n - 1)(\alpha^2 - \rho)g(Y, U)\eta(Z)) = 0.
\end{equation}

As $(\alpha^2 - \rho) \neq 0$, by treating $U = \xi$ in (7.2), we obtain

\begin{equation}
S(Y, Z) = (n - 1)(\alpha^2 - \rho)g(Y, Z).
\end{equation}

Hence, the theorem.

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**References**

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