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Stability Of Second Order Delay Differential Equations Using Modified Algebraic Approach

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Abstract:*In this paper an algebraic approach have been developed to study the stability of second order delay differential equations. The method has its basis upon transforming the characteristic equations related to delay differential equations into an equivalent system of two algebraic equations in order to evaluate the value of* w, λ, τ *which involves in the equations. The study of stability of the delay differential equation of second order, which depends upon the value of time lag which shows that the trivial solution of the characteristic equation is stable.*

Key words: *Delay differential equations (DDEs), Characteristicequations, Time lag interval, Asymptotic stability , Second order delay differential equations, Algebraic equations.*

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I. Introduction

System of delay differential equations now occupy a place of central importance in all areas of science and particularly in the biological sciences (e.g., population dynamics and epidemiology). Delay differential equations (DDEs) are popular tools used by scientists in modeling real life systems. Time delays are natural components of the dynamic processes of biology,ecology, physiology, economics, epidemiology and mechanics and so a realistic model of these processes must include time delay. In most cases, the parameters of the DDE models are unknown in which these parameters are often have important scientific interpretations and hence it is necessary to infer their values. In most cases the solution of DDEs are so difficult to be evaluated and therefore it may be sufficient to study the stability of such solutions for increasing time without evaluating this solution explicitly, i.e, study its behavior for increasing time.

Time delays are so many encountered in various real life systems, such as electric, pneumatic and hydraulic networks, chemical processes etc, so the existence of time lags, they are presented in the state or control may cause systems instability. As a consequence, the problem of stability analysis is one of the maininterests for many researchers, since in general the time delay factors make the analysis much more complicated. Stability analysis of DDEs is particular relevant in control theory , where one cause of the delay is the finite speed. Jury E.I and Zeheb E. in 1983 (See [6]) proposed an algebraic method to obtain the relevant values of certain control gain of a multivariable feedback system to be stable, in which the basis of the method involves an algebraic solution of two equations obtained from the real and imaginary parts of the characteristic equation of the system.

In this paper, the method of Jury E.I. and Zeheb E. in composition with tau-decomposition method will be modified and used to study the stability of second order delay different equations by evaluating the value of the time lag which ensures the stability of the differential equation.

Stability of Linear Delay Differential Equations :

The system of linear differential equations with retarded and neutral deviating arguments is defined by

$$x'(t) = A_1 x(t-\tau_1) + A_2 x(t-\tau_2) + \ldots + A_m x(t-\tau_m) + B_1 x'(t-\tau_1) + B_2 x'(t-\tau_2)$$
$$+ \ldots + B_m x'(t-\tau_m)$$

$$\Rightarrow x'(t) = \sum_{j=1}^{m} A_j x(t-\tau_j) + \sum_{j=1}^{m} B_j x'(t-\tau_j) \qquad \ldots(1)$$

where $\tau_j \geq 0$ for all $j = 1,2,\ldots,m$ and $x(t) \in R^n$.

Let $\bar{x}(t)$ be any solution of (1) and substituting the transformation $z(t) = x(t) - \bar{x}(t)$ will give to

$$z(t) = \sum_{j=1}^{m} A_j z(t-\tau_j) + \sum_{j=1}^{m} B_j z'(t-\tau_j)$$

If $z(t) = 0$ is stable or unstable or asymptotically stablesolution of (2), then the same is true for every solution of (1) and hence it is sufficient to study the stability of the trivial solution of (1).

Let, $x(t) = Ve^{\lambda t}$, where V is an n-dimensional vector and λ has a complex constant value, be a solution of (1).

So $x'(t) = V\lambda e^{\lambda t}$

Substituting $x(t)$ and $x^1(t)$ in equation (1) we get

$$V\lambda e^{\lambda t} = \sum_{j=1}^{m} A_j V e^{\lambda(t-\tau_j)} + \sum_{j=1}^{m} B_j V\lambda e^{\lambda(t-\tau_j)}$$

$$\Rightarrow V\lambda e^{\lambda t} = \left[\sum_{j=1}^{m} A_j V e^{-\lambda\tau_j} + \sum_{j=1}^{m} B_j \lambda e^{-\lambda\tau_j}\right] V e^{\lambda t}$$

$$\Rightarrow \left[\lambda I - \sum_{j=1}^{m} A_j e^{-\lambda\tau_j} - \sum_{j=1}^{m} B_j \lambda e^{-\lambda\tau_j}\right] V e^{\lambda t} = 0$$

In order to get a nontrivial solution, we must have

$$\det\left[\lambda I - \sum_{j=1}^{m} A_j e^{-\lambda\tau_j} - \sum_{j=1}^{m} B_j \lambda e^{-\lambda\tau_j}\right] = 0 \qquad \ldots(3)$$

The equation(3) is called a characteristic equation, which is a polynomial of degree *m*and denoted by $P(\lambda, e^{-\lambda\tau_j})$. If the roots of equation (3) could be found out, then the stability of the system of linear delay differential equation (1) may be determined based on the following theorems :

Theorem (1), see [11] :

If all the roots of the characteristic equation (3) have negative real parts, then the trivial solution of (1) is asymptotically stable.

Theorem (2), see[11] :

If at least one root has a positive real part, then the trivial solution of (1) is unstable.

Theorem (3) , see [11] :

If there are simply purely imaginary roots and the remaining roots have negative real parts, then the trivial solution of (1) is stable.

So from the above theorems we conclude that the negativeness of the real part plays an important role for stability .

The Modified Algebraic Approach

This **:** approach consists of the evaluation of the solution of two algebraic equations resulting from the real and imaginary parts of the complex characteristic equation or polynomial related to the delay differential equation. This approach which is originally established for determining the gain of multivariable feedback control systems that will be modified and improved here to study the stability of the system of second order delay differential equations by evaluating the value of the time lags which ensure the stability of the system under consideration.

Let

$$x''(t) + x'(t) = f\left(x(t), x(t-\tau), x(t-2\tau), \ldots, x(t-m\tau)\right) \ldots (4)$$

be the system of retarded delay differential equations, where $t, \tau > 0$ and $m \in N$

Putting $x(t) = V\ e^{\lambda t}\ , \lambda \in C$

The characteristic equation related to equation (4) is given by

$$P(\lambda^2, \lambda, \tau) = 0 \qquad \ldots (5)$$

So, in order to use the utilities of the complex number,

let $x(t) = V\, e^{\lambda t}$, with the assumption that λ is purely imaginary i.e, $\lambda = iw, w \in R$.

Then, the characteristic equation (5) may be written as in the following complex function :

$$P(iw, \tau) = \text{Re}[P(w, \tau)] + i\,\text{Im}[P(w, \tau)] = 0$$

$\Rightarrow \quad \mathrm{Re}\left[P(w,\tau)\right]=0$ … (6)

$\mathrm{Im}\left[P(w,\tau)\right]=0$ … (7)

Equations (6) and (7) represent an algebraic system of two equations with two variables w and τ which may be solved to find them to ensure the stability of the system of DDEs given by (4).

Since the procedure of finding the characteristic roots of (5) or equivalently (6) and (7) is not simple, as equation (5) is an exponential polynomials. Thus, to check the direction of the root loci to determine whether the roots cross from right half plane to left half plane or left half plane to right half plane.

The characteristic equation in case of delay system contains a term $e^{-\lambda\tau}$, which renders the system of algebraic equations (6) and (7) which are difficult to be solved. So, to find the value of τ, we introduce a new variable T by writing $e^{-\lambda\tau}=\dfrac{(1-\lambda T)^2}{(1+\lambda T)^2}$ as $e^{-\lambda\tau}$ is positive and less than 1.

Now we can rewrite the characteristic equation (5) as

$$P(\lambda,T)=a_0(T)\lambda^j+a_1(T)\lambda^{j-1}+\ldots+a_j(T) \quad \ldots (8)$$

where $a_0, a_1,\ldots, a_j$ are polynomials in T and it is assumed that P is irreducible i.e, $a_0(T)\neq 0$, $a_j(T)\neq 0$

Setting $\lambda = iw$, then equation (8) becomes

$$P(iw,T)=a_0(T)(iw)^j+a_1(T)(iw)^{j-1}+a_2(T)(iw)^{j-2}+a_3(T)(iw)^{j-3}+\ldots+a_{j-1}(T)iw+a_j(T) \quad \ldots (9)$$

If j is even , then equating real and imaginary parts of (9) gives

$$P_1(w,T)=a_0(T)w^j-a_2(T)w^{j-2}+\ldots+a_j(T) \quad \ldots(10)$$

$$P_2(w,T)=a_1(T)w^{j-1}-a_3(T)w^{j-3}+\ldots+a_{j-1}(T)w \quad \ldots (11)$$

If j is odd , then equating real and imaginary parts of (9) gives

$$P_1(w,T)=a_1(T)w^{j-1}-a_3(T)w^{j-3}+\ldots+a_{j-1}(T)w \quad \ldots (12)$$

$$P_2(w,T)=a_0(T)w^j-a_2(T)w^{j-2}+\ldots+a_j(T) \quad \ldots(13)$$

Necessary condition for stable solutions of second order delay differential equations :

Theorem (4) :

If $P_1(w,T)$ and $P_2(w,T)$ have no real solutions, then equation (4), i.e, $x''(t)+x'(t)=f\left(x(t),x(t-\tau),x(t-2\tau),\ldots x(t-m\tau)\right)$ is stable for all real values of the delay arguments.

Proof:

As $p_1(w,T)$ and $p_2(w,T)$ are the real and imaginary parts of (4) i.e,

$x^{11}(t)+x^{1}(t)=f\left(x(t),x(t-\tau),x(t-2\tau),\ldots,x(t-m\tau)\right)$ with $\lambda=iw$, with real finite solutions T and w and let the solution for T to be the set $\{T_1,T_2,\ldots,T_m\}$ with $T_1<T_2<\ldots<T_m$ and equation (4) is satisfied for real w are those values of T for which at least one branch of the root locus of the resulting algebraic polynomial in T intersects (or touches) the imaginary axis of the w-plane.

Since the roots of a polynomial are continuous function of its coefficients, so that each branch of the root locus is a continuous curve and a root can possibly move from the right half of the w-plane to the left half of the w-plane or vice versa, only for values of T from the set $\{T_1,T_2,\ldots,T_m\}$.

Hence the complete intervals between the successive values $T_i, i=1,2,\ldots,m$ reaches either a stable or unstable system for certain values of τ .

Thus no real solution to the system (10) and (11) i.e, $P_1(w,T)$ and $P_2(w,T)$ exists, which means that none of the branches of the algebraic equations of T for which $P(w,T)=0$ crosses or touches the imaginary axis of the w-plane.

Therefore, if the system is stable for one value of T i.e, its eigen values are clustered in the open left half of the w-plane and it will remains stable for all values of T.

Procedure to analyze the stability of retarded second order delay systems and then improved for neutral systems with single delay of modified Algebraic approach :

1. Substitute $\lambda = iw$ in equation (4) and then evaluate the algebraic system (6) and (7) with $e^{-\lambda\tau} = \frac{(1-\lambda T)^2}{(1+\lambda T)^2}$ that is equivalently reduced to

$$\left.\begin{array}{l} \mathrm{Re}[P(w,T)] = 0 \\ \mathrm{Im}[P(w,T)] = 0 \end{array}\right\} \quad \ldots (14)$$

2. Solve the resulting system (4), if possible, for non-negative real values of w. The solutions may be denoted by the ascending set $\{T_1, T_2 ..., T_m\}$
3. From in properties of the field of real numbers, the values of $T_1, T_2 ..., T_m$ divide the real line T into m+1 intervals $(T_i, T_{i+1}), i = 1,2,..., m$
4. Choose an arbitrary value in each interval where these value may be denoted by $T^{(0)}, T^{(1)}, ..., T^{(m)}$.
5. Check the zeros or roots of $P(w, T^{(i)})$, i=0,1,…, m for stability.
6. If all the roots of the characteristic equation (5) have negative real parts , then the trivial solution of (4) is asymptotically stable.
7. If at least one root of the characteristic equation (5) has a positive real part, then the trivial solution of (4) is unstable.
8. If there are simple purely imaginary roots and the remaining roots have negative real parts, then the trivial solution of (4) is stable.

Example -1

Let the retarded argument DDE with τ, where $\tau > 0$ is

$$x''(t) + x''(t) + x(t) + x(t-\tau) = 0, \quad t > 0$$

Find the value of τ for which the given second order DDE is stable.

Solution :

Given the second order DDE as

$$x''(t) + x'(t) + x(t) + x(t-\tau) = 0, \quad t > 0 \quad \ldots (15)$$

Let $\quad x(t) = Ve^{\lambda t}, \lambda \in c$

$$\Rightarrow \quad x'(t) = V\lambda e^{\lambda t},\ x''(t) = V\lambda^2 e^{\lambda t}, x(t-\tau) = Ve^{\lambda(t-\tau)}$$

So, equation (15) reduces to

$$V\lambda^2 e^{\lambda t} + V\lambda e^{\lambda t} + Ve^{\lambda t} + Ve^{\lambda(t-\tau)} = 0$$

$$\Rightarrow \quad Ve^{\lambda t}\left(\lambda^2 + \lambda + 1 + e^{-\lambda\tau}\right) = 0$$

$$\Rightarrow \quad \lambda^2 + \lambda + 1 + e^{-\lambda\tau} = 0 \quad \ldots (16)$$

which is the characteristic equation.

Letting $\lambda = iw$ (purely imaginary zeros), then equation (16) becomes

$$i^2 w^2 + iw + 1 + e^{-iw\tau} = 0$$

$$\Rightarrow \quad -w^2 + iw + 1 + \cos(w\tau) - i\sin(w\tau) = 0, \quad (\because e^{i\theta} = \cos\theta + i\sin\theta)$$

$$\Rightarrow \quad \left[1 - w^2 + \cos(w\tau)\right] + i\left[w - \sin(w\tau)\right] = 0,$$

which is an algebraic equation.

Equating real and imaginary parts, the equivalent algebraic system in w and τ are given by

$$1 - w^2 + \cos(w\tau) = 0$$

$$w - \sin(w\tau) = 0$$

The non-linear system may be solved to find w and τ or by letting $e^{-\lambda\tau} = \frac{(1-\lambda T)^2}{(1+\lambda T)^2}$, where T is a new variable. The characteristic equation (16) becomes $\lambda^2 + \lambda + 1 + \frac{(1-\lambda T)^2}{(1+\lambda T)^2} = 0$

$$\Rightarrow \lambda^2(1+\lambda T)^2 + \lambda(1+\lambda T)^2 + (1+\lambda T)^2 + (1-\lambda T)^2 = 0$$

$$\Rightarrow \lambda^2\left(1 + \lambda^2 T^2 + 2\lambda T\right) + \lambda\left(1 + \lambda^2 T + 2\lambda T\right) + \left(1 + \lambda^2 T^2 + 2\lambda T\right) + \left(1 + \lambda^2 T^2 - 2\lambda T\right) = 0$$

$\Rightarrow T^2\lambda^4 + (2T + T^2)\lambda^3 + (1 + 2T + 2T^2)\lambda^2 + \lambda + 2 = 0$

or

$P(\lambda, T) = T^2\lambda^4 + (2T + T^2)\lambda^3 + (1 + 2T + 2T^2)\lambda^2 + \lambda + 2 = 0$ … (17)

Setting $\lambda = iw$, the equation (17) becomes,

$$T^2(iw)^4 + (2T + T^2)(iw)^3 + (1 + 2T + 2\mathrm{T}^2)(iw)^2 + iw + 2 = 0$$

$$\Rightarrow \quad T^2w^4 - i(2T + T^2)w^3 - (1 + 2T + 2\mathrm{T}^2)w^2 + iw + 2 = 0$$

$$\Rightarrow \quad [T^2w^4 - (1 + 2T + 2T^2)w^2 + 2] + i[w - (2T + T^2)w^3] = 0$$

Equating real and imaginary parts, the following system of algebraic equations will be obtained:

$T^2w^4 - (1 + 2T + 2T^2)w^2 + 2 = 0$ … (18)

$w - (2T + T^2)w^3 = 0$ … (19)

From equation (19),

$$w\{1 - (2T + T^2)\ w^2\} = 0$$

$$\Rightarrow \quad w = 0 \ \text{ or } 1 - (2T + T^2)w^2 = 0$$

As $w \neq 0$ $\left(\because \textit{If} \;\; w = 0, \;\; \lambda = iw \;\;\; \textit{is not purely imaginary}\right)$

So, $1 - (2T + T^2)w^2 = 0$

$$\Rightarrow \quad w^2 = \frac{1}{2T + T^2}$$

Substituting the value of w^2 in equation (18), we get

$$T^2\left(\frac{1}{2T + T^2}\right)^2 - (1 + 2T + 2T^2)\frac{1}{(2T + T^2)} + 2 = 0$$

$$\Rightarrow \quad T^2\left(\frac{1}{T^2(T + 2)^2}\right) - \frac{(1 + 2T + 2T^2)}{T(T + 2)} + 2 = 0$$

$$\Rightarrow \quad \frac{1}{(T + 2)^2} - \frac{(1 + 2T + 2T^2)}{T(T + 2)} + 2 = 0$$

$$\Rightarrow \quad T - (T + 2)(1 + 2T + 2T^2) + 2T(T + 2)^2 = 0$$

$$\Rightarrow \quad 2T^2 + 4T - 2 = 0$$

$$\Rightarrow \quad T^2 + 2T - 1 = 0$$

$$\Rightarrow \quad T = \frac{-2 \pm \sqrt{4 + 4}}{2}$$

$$\Rightarrow \quad T = -1 \pm \sqrt{2}$$

$\Rightarrow \quad T_1 = -1 + \sqrt{2}$ (positive), $\quad T_2 = -1 - \sqrt{2}$ (Negative)

As non-negative value of T will be used, so we take $T_1 = \sqrt{2} - 1$ i.e, $T_1 = 0.4142$

when $T_1 = 0.4142$, the value of w i.e, $w_1 = \dfrac{1}{\sqrt{2(0.4142) + (0.4142)^2}} = 1.00001$

Now, the value of $T_1 = 0.4142$ will divide the non-negative real axis into two intervals $(0, T_1)$ and (T_1, ∞) i.e, $(0, 0.4142)$ and $(0.4142, \infty)$.

Therefore, we may choose an arbitrary values over each interval , say $T_1^{(0)} = 0.1$ and $T_1^{(1)} = 1$

Finally, substituting $T_1^{(0)} = 1$ in the characteristic equation (17) will give

$P(\lambda, T_1^{(0)}) = 0.01\ \lambda^4 + 0.21\lambda^3 + 1.22\lambda^2 + \lambda + 2 = 0$

which has the roots

$$\lambda = \begin{pmatrix} -10.213 - 2.029i \\ -10.213 + 2.029i \\ -0.287 - 1.327i \\ -0.287 + 1.327i \end{pmatrix} \text{(Using root finder polynomial calculator)}$$

and it is clear that all the roots of λ have negative real parts.

Again , while substituting $T_1^{(1)} = 1$ in the characteristic equation (17) will give

$$P(\lambda, T_1^{(1)}) = \lambda^4 + 3\lambda^3 + 5\lambda^2 + \lambda + 2 = 0$$

which has the roots

$$\lambda = \begin{pmatrix} -1.53 - 1.556i \\ -1.53 + 1.556i \\ 0.03 - 0.647i \\ 0.03 - 0.647i \end{pmatrix} \qquad \text{(Using root finder polynomial calculator)}$$

and it is clear that two roots have positive real parts.

As a result, the corresponding value of τ with $T \in (0, 0.4142)$ which stabilizes the retarded DDE (15) equals to $\tau = 1.5707$ and therefore the time lag interval of stability is (0, 1.5707)

Example -2

Let the retarded argument DDE with τ , where $\tau > 0$ is

$x''(t) + x'(t) + x(t - \tau) = 0, \quad t > 0$

Find the value of τ for which the given second order DDE is stable

Solution :

Given the second order DDE as

$$x''(t) + x'(t) + x(t - \tau) = 0, \quad t > 0 \qquad \ldots (20)$$

Let $\quad x(t) = Ve^{\lambda t},\ \lambda \in C$

$\Rightarrow \quad x'(t) = Ve^{\lambda t}, x''(t) = V\lambda^2 e^{\lambda t},\ x(t-\tau) = Ve^{\lambda(t-\tau)}$

So equation (20) reduces to

$V\lambda^2 e^{\lambda t} + V\lambda e^{\lambda t} + Ve^{\lambda(t-\tau)} = 0$

$\Rightarrow \quad Ve^{\lambda t}\left(\lambda^2 + \lambda + e^{-\lambda\tau}\right) = 0$

$$\Rightarrow \quad \lambda^2 + \lambda + e^{-\lambda\tau} = 0 \qquad \ldots (21)$$

which is the characteristic equation.

Letting $\lambda = iw$ (purely imaginary zeros) , then equation (21) reduces to

$i^2 w^2 + iw + e^{-iw\tau} = 0$

$\Rightarrow \quad -w^2 + iw + \cos(w\tau) - i\sin(w\tau) = 0, \left(\because e^{i\theta} = \cos\theta + i\sin\theta\right)$

$\Rightarrow \quad \left[\cos(w\tau) - w^2\right] + i\left[w - \sin(w\tau)\right] = 0$

which is an algebraic equation.

Equating real and imaginary parts, the equivalent algebraic system in w and τ are given by

$$\cos(w\tau) - w^2 = 0$$
$$w - \sin(w\tau) = 0$$

The non-linear system may be solved to find w and τ or by letting $e^{-\lambda\tau} = \dfrac{(1-\lambda T)^2}{(1+\lambda T)^2}$, where T is a new variable.

The characteristic equation (21) becomes,

$$\lambda^2+\lambda+\frac{(1-\lambda T)^2}{(1+\lambda T)^2}=0$$

$\Rightarrow \quad \lambda^2(1+\lambda T)^2+\lambda(1+\lambda T)^2+(1-\lambda T)^2=0$

$\Rightarrow \quad \lambda^2(1+\lambda^2 T^2+2\lambda T)+\lambda(1+\lambda^2 T^2+2\lambda T)+(1+\lambda^2 T^2-2\lambda T)=0$

$\Rightarrow \quad T^2\lambda^4+(2T+T^2)\lambda^3+(1+2T+T^2)\lambda^2+(1-2T)\lambda+1=0$

or

$$P(\lambda,T)=T^2\lambda^4+\left(2T+T^2\right)\lambda^3+\left(1+2T+T^2\right)\lambda^2+\left(1-2T\right)\lambda+1=0 \qquad \ldots(22)$$

Setting $\lambda=iw$, the equation (22) becomes

$$T^2(iw)^4+\left(2T+T^2\right)(iw)^3+\left(1+2T+T^2\right)(iw)^2+\left(1-2T\right)(iw)+1=0$$

$\Rightarrow \quad T^2w^4-i\left(2T+T^2\right)w^3-\left(1+2T+T^2\right)w^2+i\left(1-2T\right)w+1=0$

$\Rightarrow \quad \left[T^2w^4-\left(1+2T+T^2\right)w^2+1\right]+i\left[\left(1-2T\right)w-\left(2T+T^2\right)w^3\right]=0$

Equating real and imaginary parts, the following systemof algebraic equations will be obtained:

$$T^2w^4-(1+2T+T^2)w^2+1=0 \qquad \ldots (23)$$

$$(1-2T)w-(2T+T^2)w^3=0 \qquad \ldots (24)$$

From equation (24),

$$w\left\{(1-2T)-(2T+T^2)w^2\right\}=0$$

$\Rightarrow \quad w=0$ or $(1-2T)-(2T+T^2)w^2=0$

As $w\neq 0$ ($\because$ If $w=0$, $\lambda=iw$ is not purely imaginary)

So $(1-2T)-(2T+T^2)w^2=0$

$\Rightarrow \quad w^2=\dfrac{1-2T}{2T+T^2}$

$\Rightarrow \quad w=\sqrt{\dfrac{1-2T}{2T+T^2}}$

Substituting the value of w in equation (23) we get

$$T^2\left(\frac{1-2T}{2T+T^2}\right)^2-(1+2T+T^2)\left(\frac{1-2T}{2T+T^2}\right)+1=0$$

$\Rightarrow \quad T^2\dfrac{(1-2T)^2}{T^2(T+2)^2}-\dfrac{(1+2T+T^2)(1-2T)}{T(T+2)}+1=0$

$\Rightarrow \quad T(1-2T)^2-(T+2)(1+2T+T^2)\ (1-2T)+T(T+2)^2=0$

$\Rightarrow \quad 2T^4+12T^3+6T^2+4T-2=0$

$\Rightarrow \quad T^4+6T^3+3T^2+2T-1=0$

$$\Rightarrow \quad T=\begin{pmatrix}-5.529\\ 0.293\\ -0.382-0.687i\\ -0.382+0.687i\end{pmatrix} \qquad \text{(using root finder polynomial calculator)}$$

As non-negative real values of T will be used so we take $T_1=0.293$

when $T_1=0.293$, the value of w i.e, $w_1=\sqrt{\dfrac{1-2(0.293)}{2(0.293)+(0.293)^2}}=0.7849$

Now, the value of $T_1 = 0.293$ will divide the non-negative real axis into two intervals $(0, T_1)$ and (T_1, ∞) i.e, $(0,\ 0.293)$ and $(0.293, \infty)$

Therefore, we may choose an arbitrary values over each interval, say $T_1^{(0)} = 0.1$ and $T_1^{(1)} = 1$

Finally , substituting $T_1^{(0)} = 0.1$ in the characteristic equation (22) will give

$$P(\lambda, T_1^{(0)}) = 0.01\lambda^4 + 0.21\lambda^3 + 1.21\lambda^2 + 0.8\lambda + 1 = 0$$

which has the roots

$$\lambda = \begin{pmatrix} -10.218 - 2.037i \\ -10.218 + 2.037i \\ -0.282 - 0.917i \\ -0.282 + 0.917i \end{pmatrix} \qquad \text{(using root finder polynomial calculator)}$$

and it is clear that all the roots of λ have negative real parts.

Again, while substituting $T_1^{(1)} = 1$ in the characteristic equation (22) will give

$$P(\lambda, T_1^{(1)}) = \lambda^4 + 3\lambda^3 + 4\lambda^2 - \lambda + 1 = 0$$

which has the roots

$$\lambda = \begin{pmatrix} -1.67 - 1.464i \\ -1.67 + 1.464i \\ 0.17 - 0.417i \\ 0.17 + 0.417i \end{pmatrix} \qquad \text{(using root finder polynomial calculator)}$$

and it is clear that two roots have positive real parts . As a result, the corresponding value of λ with $T \in (0, 0.293)$ which stabilizes the retarded DDE (20) equals to $\lambda = 1.1556$ and therefore the time lag interval of stability is (0, 1.1556).

Example :3

Consider the retarded argument DDE with τ, where $\tau > 0$ is

$$x''(t) + x'(t) + x(t) = x(t - \tau), \quad t > 0$$

Find the value of τ for which the given second order DDE is stable.

Solution :

Given the second order DDE as

$$x''(t) + x'(t) + x(t) = x(t - \tau), \quad t > 0 \qquad \ldots (25)$$

Let $\quad x(t) = Ve^{\lambda t}, \lambda \in C$

$\Rightarrow \quad x'(t) = V\lambda e^{\lambda t},\ x''(t) = V\lambda^2 e^{\lambda t}, x(t - \tau) = Ve^{\lambda(t-\tau)}$

So equation (25) reduces to

$$V\lambda^2 e^{\lambda t} + V\lambda e^{\lambda t} + Ve^{\lambda t} = Ve^{\lambda(t-\tau)}$$

$$\Rightarrow \quad Ve^{\lambda t}(\lambda^2 + \lambda + 1) = Ve^{\lambda t} e^{-\lambda\tau}$$

$$\Rightarrow \quad \lambda^2 + \lambda + 1 - e^{-\lambda\tau} = 0 \qquad \ldots (26)$$

which is the characteristic equation

Letting $\lambda = iw$ (purely imaginary zeros), then equation (26) reduces to

$$i^2 w^2 + iw + 1 - e^{-iw\tau} = 0$$

$$\Rightarrow \quad -w^2 + iw + 1 - (\cos(w\tau) - i\sin(w\tau)) = 0$$

$$\Rightarrow \quad -w^2 + iw + 1 - \cos(w\tau) + i\sin(w\tau) = 0$$

$$\Rightarrow \quad [1 - w^2 - \cos(w\tau)] + i[w + \sin(w\tau)] = 0$$

which is an algebraic equation.

Equating real and imaginary parts, the equivalent algebraic system in w and τ are given by

$$1 - w^2 - \cos(w\tau) = 0$$

$$w + \sin(w\tau) = 0$$

The non-linear system may be solved to find w and τ or by letting $e^{-\lambda\tau} = \frac{(1-\lambda T)^2}{(1+\lambda T)^2}$, where T is a new variable.

The characteristic equation(26) becomes

$$\lambda^2 + \lambda + 1 - \frac{(1-\lambda T)^2}{(1+\lambda T)^2} = 0$$

$$\Rightarrow \quad \lambda^2(1+\lambda T)^2 + \lambda(1+\lambda T)^2 + (1+\lambda T)^2 - (1-\lambda T)^2 = 0$$

$$\Rightarrow \quad T^2\lambda^4 + (2T+T^2)\lambda^3 + (1+2T)\lambda^2 + (1+4T)\lambda = 0$$

or

$$P(\lambda, T) = T^2\lambda^4 + (2T+T^2)\lambda^3 + (1+2T)\lambda^2 + (1+4T)\lambda = 0 \quad \ldots (27)$$

Setting $\lambda = iw$, the equation (27) becomes

$$T^2(iw)^4 + (2T+T^2)(iw)^3 + (1+2T)(iw)^2 + (1+4T)(iw) = 0$$

$$\Rightarrow \quad w^4T^2 - iw^3(2T+T^2) - w^2 + (1+2T) + iw(1+4T) = 0$$

$$\Rightarrow \quad [w^4T^2 - w^2(1+2T)] + i[w(1+4T) - w^3(2T+T^2)] = 0$$

Equating real and imaginary parts , the following systemof algebraic equations will be obtained :

$$T^2w^4 - (1+2T)w^2 = 0 \quad \ldots (28)$$

$$(1+4T)w - (2T+T^2)w^3 = 0 \quad \ldots (29)$$

Solving (28), we get

$$w^2[T^2w^2 - (1+2T)] = 0$$

$$\Rightarrow \quad T^2w^2 - (1+2T) = 0 \quad (\because \quad w \neq 0)$$

$$\Rightarrow \quad w^2 = \frac{1+2T}{T^2}$$

$$\Rightarrow \quad w = \sqrt{\frac{1+2T}{T^2}}$$

Again from (29),

$$w\{(1+4T) - (2T+T^2)w^2\} = 0$$

$$\Rightarrow \quad (1+4T) - (2T+T^2)w^2 = 0 \ (\because w \neq 0)$$

$$\Rightarrow \quad (1+4T) - (2T+T^2)\left(\frac{1+2T}{T^2}\right) = 0$$

$$\Rightarrow \quad (1+4T) - \frac{(T+2)(1+2T)}{T} = 0$$

$$\Rightarrow \quad 2T^2 - 4T - 2 = 0$$

$$\Rightarrow \quad T^2 - 2T - 1 = 0$$

$$\Rightarrow \quad T = \frac{2 \pm \sqrt{8}}{2}$$

$$\Rightarrow \quad T = 1 \pm \sqrt{2}$$

$$\Rightarrow \quad T_1 = 1+\sqrt{2} \text{ (positive)}, \ T_2 = 1-\sqrt{2} \text{ (negative)}$$

As non-negative real values of T will be used, so we take

$$\Rightarrow \quad T_1 = 1+\sqrt{2}$$

$$\Rightarrow \quad T_1 = 2.4142$$

when $T_1 = 2.4142$, the value of w i.e,

$$w_1 = \sqrt{\frac{1+2(2.4142)}{(2.4142)^2}} = 1.000003$$

Now, the value of $T_1 = 2.4142$ will divide the non-negative real axis into two intervals $(0, T_1)$ and (T_1, ∞) i.e, (0,2.4142) and (2.4142, ∞)

Therefore, we may choose an arbitrary values overeach interval, say, $T_1^{(0)} = 1$ and $T_1^{(1)} = 3$

Finally, substituting $T_1^{(0)} = 1$ in the characteristic equation (27) will give

$P(\lambda, T_1^{(0)}) = \lambda^4 + 3\lambda^3 + 3\lambda^2 + 5\lambda = 0$

which has the roots

$$\lambda = \begin{pmatrix} -2.587 \\ 0 \\ -0.206 - 1.375i \\ -0.206 + 1.375i \end{pmatrix}$$

which is clear that three roots have negative real parts and one root is zero

Again, while substituting $T_1^{(1)} = 3$ in the characteristic equation (27) will give

$P(\lambda, T_1^{(1)}) = 9\lambda^4 + 15\lambda^3 + 7\lambda^2 + 13\lambda = 0$

which has the roots

$$\lambda = \begin{pmatrix} -1.707 \\ 0 \\ 0.02 - 0.92\, i \\ 0.02 + 0.92\, i \end{pmatrix}$$

whichis clear that two roots have positive real parts.

As in both cases, at least one root of the characteristic equation (26) hasa positive real parts, so the trivial solution of (25) is unstable.

Remarks:

It is to be noted that the trivial solution of (25) is unstable as $P(\lambda, T)$ in (27) is not irreducible i.e, $a_4(T) = 0$ in (27).

II. Conclusion

The stability of DDEs may be affected by the existence of time lags. Therefore the evaluations of the time lags seems to be necessary to study the stability of DDE. Also the value of w, τ which stabilizes the system depends on the real life system under consideration. In the present approach, evaluation of time lag, the solution of the characteristic equation is stable due to the presence of negative real parts of all the roots.

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