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Supereulerian and Trailable Digraph Products

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Abstract: *A digraph D is **supereulerian** if D contains a spanning eulerian subdigraph, and is **trailable** if D contains a spanning ditrail. Sufficient conditions on D_1 and D_2 are obtained for the Cartesian product digraph and Lexicographic product digraph of D_1 and D_2 to be supereulerian or trailable.*

***Key words**. Combinatorial problems, Supereulerian digraph, Cartesian product, Lexicographic product, Eulerian digraph*

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I. Introduction

We consider finite graphs and digraphs.Undefined terms and notation will follow [6] for graphs and [3] fordigraphs.Wewrite$D_1 \cong D_2$to

Denote that the digraphs D_1 and D_2 are is omorphic. A sin[6], uv represents an edge joining uandv. Asin[3],a digraph does not have **parallel**arcs, that is, pairs of arcs with the same tailand the same head, or **loops**.TheunderlyinggraphofadigraphD,denotedbytt(D),isobtainedfromDbyerasingtheorientationsofallarcsofD.Throughoutthispaper,weusethenotation(u,v)todenoteanarcorientedfromutovinadigraph;anduse[u,v]todenoteanarcwhichiseither(u,v)or(v,u).Foranintegern,wedefine[n]={1,2,···,n}.A**walk**inDisanalternatingsequence W = $x_1a_1x_2a_2x_3 \cdot \cdot \cdot x_{k-1}a_{k-1}x_k$of vertices $x_i$and arcs $a_j$from D such that $a_j = (x_j, x_{j+1})$ for every i∈[k]andj∈[k−1].AwalkWis**closed**if$x_1=x_k$,and**open**otherwise.WeuseV(W)=$\{x_i{:}i\in[k]\}$ andA(W)=$\{a_j{:}j\in[k-1]\}$.WesaythatWisawalkfrom$x_1$to$x_k$oran(x_1,x_k)-walk.If$x_1 f = x_k$, then wesaythatthevertex$x_1$isthe**initailvertex**ofW,thevertex$x_k$isthe**terminalvertex**ofW, and $x_1$and $x_k$are end-vertices of W . The length of a walk is the number of its arcs. When the arcsof W are understood from the context, we will denote W by $x_1x_2x_k$. A **ditrail** in D is a walk in which allarcsaredistinct.Alwaysweuseaditrailtodenoteanopenditrail.IftheverticesofWaredistinct, then W is a **dipath**. If the vertices $x_1x_2x_{k-1}$are distinct, k ““ 3 and $x_1= x_k$, then W is a **dicycle**. A digraph D is **strong** if, for every pair x, y of distinct vertices in D, there exist an (x, y)-walk and a (y,x)-walk.AdigraphD is**weaklyconnected**iftt(D)isconnected.IfX⊆V(D)∪A(D),thenD(X) denotes the subdigraph induced by X. For a digraph D and a set B ⊆ A(D), the digraph D − B is the spanning subdigraph of D with arc set A(D) − B. If H is a subdigraph of D and S ⊆ A(D) − A(H) withV(D(S))⊆V(H),thedigraphH+SisthesubdigraphofDwitharcsetA(H)+S andvertexset V(H).WeoftenwriteD−aforD−{a}andD+aforD+{a}.LetD_1andD_2betwodigraphs,the **union**$D_1 \cup D_2$ofD_1andD_2isadigraphwithvertexsetV($D_1 \cup D_2$)= V(D_1)∪V(D_2)andarcset A($D_1 \cup D_2$)=A(D_1)∪A(D_2).

Following [3], for X, Y ⊆ V (D), define

$$(X, Y)_D = \{(x, y) \in A(D){:}\, x \in X, y \in Y\}.$$

For a vertex v in D, we use the following notation:

$$N_D^+(v) = \{u \in V(D) - v{:}\,(v, u) \in A(D)\}, N_D^-(v) = \{w \in V(D) - v{:}\,(w, v) \in A(D)\}.$$

The sets $N_D^+(v), N_D^-(v)$ and $N_D(v) = N_D^+(v) \cup N_D^-(v)$ are called the **out-neighbourhood**, **in-neighbourhood** and **neighbourhood** of v. We called the vertices in $N_D^+(v), N_D^-(v)$ and $N_D(v)$ the **out-neighbours**, **in-neighbours** and **neighbours** of v.
neighbours and **neighbours** of v.

For a set $X \subseteq V(D)$, $d_D^+(X) = |(X, V(D) - X)_D|$ is the **out-degree** of X and $d_D^-(X) = |(V(D) - X, X)_D|$ is the **in-degree** of X. The degree of X is the number $d_D(X) = d_D^+(X) + d_D^-(X)$. When the digraph D is understood from the context, we often omit the subscript D.
Next, we use the following definitions of Cartesian product and Lexicographic product of digraphs [3].

Next, we use the following definitions of Cartesian product and Lexicographic product of digraphs [3].
Deftnition 1.1 Let $D_1= (V_1, A_1)$ and $D_2= (V_2, A_2)$ be two digraphs, $V_1= \{u_1, u_2, \cdots, u_{n1}\}$, $V_2= v_1, v_2, \{ \dots, v_{n2}$. Then the Cartesian product andLexicographic product of $D_1$and $D_2$are defined as following
(i) **The Cartesian product** denoted by $D_1\times D_2$is the digraph with vertex set $V_1\times V_2$and
$A(D_1\times D_2) = \{((u_i, v_j), (u_s, v_t)) : u_i= u_s \text{ and } (v_j, v_t) \in A_2, \text{ or } (u_i, u_s) \in A_1 \text{and } v_j= v_t\}$.
(ii) **TheLexicographicproduct**denotedby$D_1[D_2]$isthedigraphwithvertexset$V_1\times V_2$and
$A(D_1[D_2]) = \{((u_i, v_j), (u_s, v_t)) : u_i= u_s \text{ and } (v_j, v_t) \in A_2, \text{ or } (u_i, u_s) \in A_1\}$.

Boesch, Suffel, and Tindell [5] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs; and they indicated that such this problem would be very difficult. Pulleyblank [14] in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. As of today, there have been lots of researches on it. See Catlin's survey [7] and the updates in [8] and [13] for a liternature in the topic.

It is natural to study supereulerian digraphs. A digraph D is **eulerian** if $G(D)$ is connected and for every $v \in V(D)$, $d_D^+(v) = d_D^-(v)$; and is **supereulerian** if D contains a spanning eulerian subdigraph; and is **trailable** if D contains a spanning ditrail. Earlier studies were done by Gutin [10, 11]. Recent developments can be found in [2, 4, 12], among others.

In [9], an open problem (Problem 6 of [9]) was raised to find natural conditions for the product of graphs to be hamiltonian. Motivated by this problem, we propose to seek natural conditions on digraphs D_1 and D_2 such that the product of D_1 and D_2 is supereulerian. In this paper, sufficient conditions on D_1 and D_2 for $D_1 \times D_2$ and $D_1[D_2]$ to be supereulerian or trailable are investigated.

II. MainResults

2.1 Notations

The following notation will be used througout this section. Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs with $V_1 = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \cdots, v_{n_2}\}$. For each fixed $v_j \in V_2$, define $D_1^{v_j}$ to be the digraph with vertex set $V_1^{v_j} = \{(u_i, v_j)\text{: for any } u_i \in V_1\}$, and arc set $A_1^{v_j} = \{((u_i, v_j), (u_s, v_j)): (u_i, u_s) \in A_1\}$. Similarly, for each fixed $u_i \in V_1$, define $D_2^{u_i}$ to be the digraph with vertex set $V_2^{u_i} = \{(u_i, v_j)\text{: for any } v_j \in V_2\}$, and arc set $A_2^{u_i} = \{((u_i, v_j), (u_i, v_t)): (v_j, v_t) \in A_2\}$. The following observations are immidiate:

Observation 2.1 Each of the following holds.
(i) $D_1^{v_j}$, $D_2^{u_i}$ are subdigraphs of $D_1 \times D_2$ and $D_1[D_2]$, and $D_1^{v_j} \cong D_1$, $D_2^{u_i} \cong D_2$ for any $i \in [n_2]$, and for any $j \in [n_1]$.
(ii) $V(D_1 \times D_2) = V(D_1[D_2]) = \bigcup_{j=1}^{n_2} V(D_1^{v_j}) = \bigcup_{i=1}^{n_1} V(D_2^{u_i})$.
(iii) $V(D_1^{v_j}) \cap V(D_1^{v_t}) = \emptyset, \text{if} v_j, v_t \in V_2 \text{and} v_j \neq v_t$; $V(D_2^{u_i}) \cap V(D_2^{u_s}) = \emptyset, \text{if} u_i, u_s \in V_1 \text{and} u_i \neq u_s$.
(iv) $V(D_1^{v_j}) \cap V(D_2^{u_i}) = \{(u_i, v_j)\} \text{and} A(D_1^{v_j}) \cap A(D_2^{u_i}) = \emptyset \text{for} u_i \in V_1, v_j \in V_2$.

For any subdigraph $H_1 \subseteq D_1$ and $v \in V_2$, we use H_1^v to denote the subdigraphs of D_1^v with $V(H_1^v) = \{(u_i, v): u_i \in V(H_1)\}$ and $A(H_1^v) = \{((u_i, v), (u_s, v)): (u_i, u_s) \in A(H_1)\}$. Similarly, for any subdigraph $H_2 \subseteq D_2$ and $u \in V_1$, we use H_2^u to denote the subdigraphs of D_2^u with $V(H_2^v) = \{(u, v_i): v_i \in V(H_2)\}$ and $A(H_2^v) = \{((u, v_i), (u, v_s)): (v_i, v_s) \in A(H_2)\}$.

2.2 Cartesian product ofdigraphs

Sufficient conditions will be investigated in this section for the Cartesian product of $D_1$and $D_2$to be supereulerian or trailable. The results below are useful.
Theorem 2.1 (J.M. Xu [15]) Let D_1 and D_2 be eulerian digraphs. Then the Cartesian product $D_1 \times D_2$ is eulerian.

Lemma 2.1 (K.A. Alsatami et al, Lemma 2 of [1]) A digraph D is nonsupereulerian if for some integer $m > 0$, $V(D)$ has vertex-disjoint subsets $B, B_1, \cdots, B_m$ satisfying both of the following:
(i) $N^-(B_i) \subseteq B$, for $i \in [m]$.

(ii) $|\partial^-(B)| \leq m-1$.

Lemma 2.1 can be applied to find examples for digraph D to be nonsupereulerian. In the following, we present some tools needed in our arguments.

Definition 2.1 Let D be a digraph, $F_1, F_2, \cdots, F_k$ be eulerian subdigraphs of D, and let $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$.
(i) $\mathcal{F}$ is called an **eulerian vertex cover** of D, if $V(D) = \cup_{F_i \in \mathcal{F}} V(F_i)$ and $F = \cup_{F_i \in \mathcal{F}} F_i$ is weakly connected.
(ii) For any $u, v \in V(D)$, $\mathcal{F}$ is called an **eulerian chain** joining u and v, if $u \in V(F_1)$, $v \in V(F_k)$, and $V(F_i) \cap V(F_{i+1}) \neq \emptyset$ for every $i \in [k-1]$.
In [3], a digraph D is called **cyclically connected** if for every pair x, y of distinct vertices of D there is a sequence of dicycles $C_1, C_2, \cdots, C_k$ such that x is in C_1, y is in C_k, and C_i and C_{i+1} have at least one common vertex for every $i \in [k-1]$. The following theorem are useful.
Theorem 2.2 [3] A digraph D is strong if and only if it is cyclically connected.

Proposition 2.1 Let D be a weakly connected digraph. Then the following are equivalent.
(i) D is strong.
(ii) D is cyclically connected.
(iii) $\forall u, v \in V(D)$, D has an eulerian chain joining u and v.
(iv) D has an eulerian vertex cover.

Proof.(i) ⇔ (ii). By Theorem 2.2, the result is hold.
(ii) ⇒ (iii). As dicycles are eulerian digraphs, every dicycle sequence is also joining u and v, and is also an eulerian chain.
(iii) ⇒ (iv). We may assume that $|V(D)| \geq 2$. By (iii), D has an eulerian subdigraph. By Definition 2.1, every eulerian subdigraph has an eulerian vertex cover. Let D' be a subdigraph of D such that D' has an eulerian vertex cover $\mathcal{F}'$ with $|V(D')|$ maximal. If $V(D') = V(D)$, then done. Assume that $|V(D')| < |V(D)|$. Then there exist $u \in V(D) - V(D')$ and $v \in V(D')$. By (iii), D has an eulerian chain $\mathcal{F}_1 = \{F_1, F_2, \cdots, F_k\}$ joining u and v. By Definition 2.1, $D' \cup D[\cup_{i=1}^{k} A(F_i)]$ is also a subdigraph with an eulerian vertex cover $\mathcal{F}' \cup \mathcal{F}_1$, contrary to the maximality of D'. Hence (iv) must hold.
$(iv) \Rightarrow (i)$ Let D' be a maximal strong component of D. If $V(D') = V(D)$, then (i) holds. Otherwise $\exists u \in V(D')$ and $v \in V(D) - V(D')$. By (iv), D has an eulerian vertex cover $\mathcal{F} = \{F_1, F_2, \cdots, F_k\}$. Since $\mathcal{F}$ is weakly connected, there exists an $F_i \in \mathcal{F}$ with $V(F_i) \cap V(D) \neq \emptyset$ and $V(F_i) - V(D') \neq \emptyset$. It follows by definition that $D[A(D') \cup A(F_i)]$ is strong, contrary to the maximality of D'.
In the following, we will show some sufficient conditions on D_1 and D_2 to assure that the Cartesian product $D_1 \times D_2$ is supereulerian or trailable.

Theorem 2.3 Let D_1 and D_2 be two strong digraphs with $min\{|V(D_1)|, |V(D_2)|\} \geq 2$ such that D_1 is supereulerian and D_2 has an eulerian vertex cover with m eulerian subdigraphs such that $m \leq |V(D_1)|$. Then the Cartesian product $D_1 \times D_2$ is supereulerian.

Proof. Let $V(D_1) = \{u_1, u_2, \cdots, u_m, u_{m+1}, \cdots, u_{n_1}\}$ and $V(D_2) = \{v_1, v_2, \cdots, v_{n_2}\}$. Let $\mathcal{F} = \{F_1, F_2, \cdots, F_m\}$ be an eulerian vertex cover of D_2. Since D_1 is a supereulerian digraph, D_1 has a spanning eulerian ditrail H_1. By Observation 2.1, let

$$H = (\cup_{j=1}^{n_2} H_1^{v_j}) \cup (\cup_{i=1}^{m} F_i^{u_i}).$$

We want to prove that H is a spanning eulerian subdigraph of $D_1 \times D_2$. Since H_1 is a spanning eulerian ditrail of D_1, so by Observation 2.1 (i), (ii) and (iii),

$$\cup_{j=1}^{n_2} V(H_1^{v_j}) = V(D_1 \times D_2),\ and\ for\ any\ v_j, v_t \in V(D_2)\ if\ v_j \neq v_t,\ then\ V(H_1^{v_j}) \cap V(H_1^{v_t}) = \emptyset.$$

Hence H is a spanning subdigraph. In the following, we will show that $d_H^+((u_i, v_j)) = d_H^-((u_i, v_j))$ for all $(u_i, v_j) \in V(H)$.
By Observation 2.1 (iii) and (iv),

$$(\cup_{j=1}^{n_2} A(H_1^{v_j})) \cap (\cup_{i=1}^{m} A(F_i^{u_i})) = \emptyset. \qquad (1)$$

Since H_1 is a spanning eulerian ditrail of D_1, it follows that $d_{H_1}^+(u_i) = d_{H_1}^-(u_i)\ \ for\ \ u_i \in V(H_1)$. And by (1), we get that

$$d_{H_1^{v_j}}^+((u_i, v_j)) = d_{H_1^{v_j}}^-((u_i, v_j))\ \ for\ all\ \ (u_i, v_j) \in V(H_1^{v_j}).(2)$$

Since F_i is an eulerian subdigraph in D_2 for $i \in [m]$, we get that $d_{F_i}^+(v_j) = d_{F_i}^-(v_j)\ \ for\ \ v_j \in V(F_i)$. By

Observation 2.1 (iii),

$$V(F_s^{u_s}) \cap V(F_h^{u_h}) = \emptyset \quad for \quad s,h \in [m] \quad and \quad s \neq h. \ (3)$$

By (1) and (3), we get that

$$d^+_{F_i^{u_i}}\left((u_i, v_j)\right) = d^-_{F_i^{u_i}}\left((u_i, v_j)\right) \ \ for \ \ all \ \ (u_i, v_j) \in V(F_i^{u_i}) (4)$$

Thus, by (2) and (4), we get that $d^+_H((u_i, v_j)) = d^-_H((u_i, v_j))$ for all $(u_i, v_j) \in V(H)$.

Now, we prove that for any two distinct vertices (u_i, v_s) and (u_j, v_t) in $V(H)$, there is a $((u_i, v_s), (u_j, v_t))$-dipath in H. By Proposition 2.1, there exists an eulerian chain $F' = \{F_{i_1}, F_{i_2}, \cdots, F_{i_h}\}$ joining v_s and v_t in D_2 such that $v_s \in V(F_{i_1})$ and $v_t \in V(F_{i_h})$. Let $F_{i_l}^{u_{i_l}} \cong F_{i_l}$ be the subdigraph of $D_2^{u_{i_l}}$ at the fixed vertex u_{i_l}, where $i_l \in [m]$ for $l \in [h]$. By the definition of an eulerian chain, $V(F_{i_{l-1}}) \cap V(F_{i_l}) \neq \ \emptyset$, pick a vertex $v_{(l-1,l)}$ in $V(F_{i_{l-1}}) \cap V(F_{i_l})$ for $l \in \{2,3,\cdots,h\}$. Let $u_{i_1} = u_i$, $u_{i_h} = u_j$, $v_{(0,1)} = v_s$ and $v_{(h,h+1)} = v_t$, and let $P_{F_{i_l}}$ be the $((u_{i_l}, v_{(l-1,l)}), (u_{i_l}, v_{(l,l+1)}))$-dipath in $F_{i_l}^{u_{i_l}}$ and $P_{i_{(l-1,l)}}^{v_{(l-1,l)}}$ be the $((u_{i_{l-1}}, v_{(l-1,l)}), (u_{i_l}, v_{(l-1,l)}))$-dipath in $H_1^{v_{(l-1,l)}}$. Thus,

$$P = (\cup_{l=1}^{h} P_{F_{i_l}}) \cup (\cup_{l=2}^{h} P_{i_{(l-1,l)}}^{v_{(l-1,l)}})$$

is a dipath from (u_i, v_s) to (u_j, v_t) in $V(H)$. This proves the Theorem.

Example 2.1 below presents a supereulerian digraph D_1 and a strong digraph D_2 which has an eulerian vertex cover with m eulerian subdigraphs, where $m > |V(D_1)|$ such that the Cartesian product $D_1 \times D_2$ is nonsupereulerian.

Example2.1 LetD$_1$beasupereuleriandigraphwithV(D$_1$)={u$_1$,u$_2$}andA(D$_1$)={(u$_1$,u$_2$),(u$_2$,u$_1$)}. LetD$_2$beastrongdigraphwithV(D$_2$)={v$_1$,v$_2$,v$_3$,v$_4$,v$_5$}andA(D$_2$)={(v$_2$,v$_1$),(v$_1$,v$_3$),(v$_3$,v$_2$),(v$_1$,v$_4$), (v$_4$, v$_2$), (v$_1$, v$_5$), (v$_5$, v$_2$)}, which has an eulerian vertex cover with 3 eulerian subdigraphs. By definition 1.1, we can obtain the Cartesian product D$_1$× D$_2$of D$_1$and D$_2$(See Figure 1). Let B, B$_1$, B$_2$and B$_3$be vertex-disjoint subsets of V (D$_1$× D$_2$) with B = {(u$_1$, v$_1$), (u$_2$, v$_1$)}, B$_1$= {(u$_1$, v$_3$), (u$_2$, v$_3$)}, B$_2$= {(u$_1$,v$_4$),(u$_2$,v$_4$)}andB$_3$={(u$_1$,v$_5$),(u$_2$,v$_5$)}.WefindthatN$^-$(B$_i$)⊆Bfori∈{1,2,3}and|∂^-(B)|= 2. By Lemma 2.1, the Cartesian product D$_1$× D$_2$is nonsupereulerian.

u$_1$

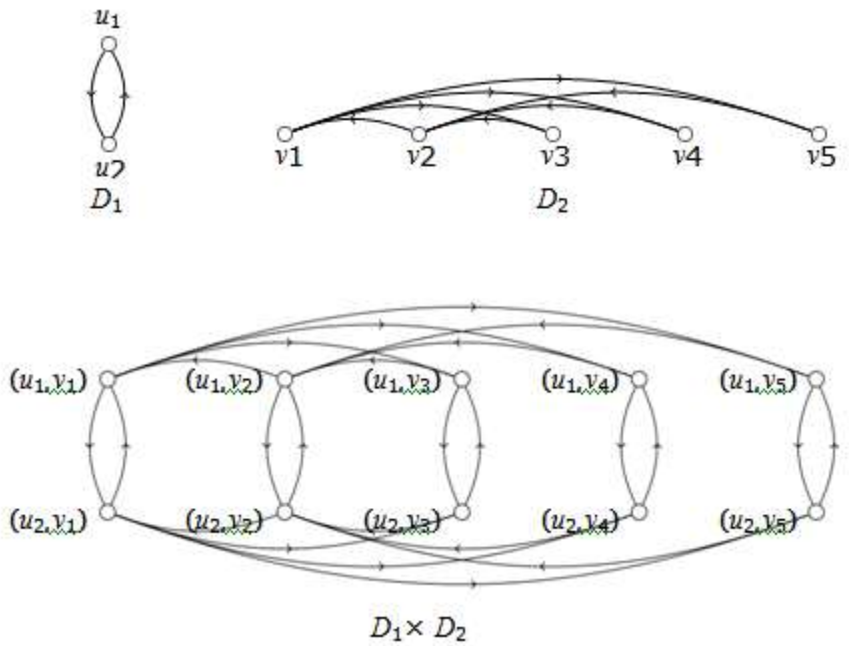

Figure 1. The digraphs D$_1$, D$_2$and the Cartesion product D$_1$× D$_2$

Example 2.1 indicates that if D$_1$is a supereulerian digraph with |V (D$_1$)| = 2 and D$_2$is a strong digraphwhichhasaneulerianvertexcoverwith3euleriansubdigraphs,thenD$_1$×D$_2$isnonsupereulerian. In fact, for n$_1$, n$_2$∈ **N** and n$_2$≥ n$_1$+ 3, the example can be extended to infinite case: Let D$_1$be a dicycle withV(D$_1$)={u$_1$,u$_2$,⋯,u$_{n1}$},letD$_2$beastrongdigraphwithV(D$_2$)={v$_1$,v$_2$,⋯,v$_{n2}$}andA(D$_2$)= {(v$_2$, v$_1$), (v$_1$, v$_3$), (v$_3$, v$_2$), (v$_1$, v$_4$), (v$_4$, v$_2$), · · · , (v$_1$, v$_{n2}$), (v$_{n2}$, v$_2$)}. D$_2$has an eulerian vertex cover with

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×

n_2-2eulerian subdigraphs$\{D[\{v_1,v_2,v_{i+2}\}]: i \in [n_2-2]\}$.Let$B,B_1,B_2,\cdots,B_{n2-2}$be vertex-disjoint subsets of V ($D_1\times D_2$) with B = {(u_1, v_1), (u_2, v_1), · · · , (u_{n1} , v_1)} and B_i= {(u_1, v_{i+2}), (u_2, v_{i+2}), · · · , (u_{n1} , v_{i+2})} for i ∈ [n_2− 2]. We find that $N^-(B_i) \subseteq B$ for i ∈ [n_2− 2] and $|\partial^-(B)| = n_1 \le n_2 - 3 = (n_2 2)\ 1$. By Lemma 2.1, the Cartesian product $D_1D_2$is nonsupereulerian. These examplesindicate
that Theorem 2.3 is best possible in some sense.
If D_2has an eulerian vertex cover with one (for m = 1 in Theorem 2.3) eulerian subdigraph, then D_2
is supereulerian. The following corollary can be obtained.

Corollary 2.1 Let D_1 be a supereulerian digraph and D_2 be a digraph.
(i) If D_2 is supereulerian, then the Cartesian product $D_1 \times D_2$ is supereulerian.
(ii) If D_2 is trailable, then the Cartesian product $D_1 \times D_2$ is trailable.

Proof. Let $V(D_1) = \{u_1, u_2, \cdots, u_{n_1}\}$, $V(D_2) = \{v_1, v_2, \cdots, v_{n_2}\}$, and let $u_{i_1} = u_1$, $v_{j_1} = v_1$. First, we will show that (i) holds. If $|V(D_1)| = 1$, then $D_1 \times D_2 \cong D_2$ is supereulerian. If $|V(D_2)| = 1$, then $D_1 \times D_2 \cong D_1$ is supereulerian. Hence we assume that $|V(D_i)| \geq 2$ for $i = 1,2$. Since D_2 is supereulerian, let $H_{21} = v_{j_1}v_{j_2}\cdots v_{j_{h_1}}v_{j_1}$ be a spanning eulerian ditrail of D_2, where $j_1, j_2, \cdots, j_{h_1} \in [n_2]$. Then H_{21} is an eulerian vertex cover with one eulerian subdigraph. Thus, (i) follows by Theorem 2.3, for $m = 1$.

Next, we will prove that (ii) holds. If $|V(D_1)| = 1$, then $D_1 \times D_2 \cong D_2$ is trailable. If $|V(D_2)| = 1$, then $D_1 \times D_2 \cong D_1$ is supereulerian, which is also trailable. Hence we assume that $|V(D_i)| \geq 2$ for $i = 1,2$. Since D_1 is supereulerian, let $H_1 = u_{i_1}u_{i_2}\cdots u_{i_{h_1}}u_{i_1}$ be a spanning eulerian ditrail of D_1, where $i_1, i_2, \cdots, i_{h_1} \in [n_1]$. Since D_2 has a spanning ditrail denoted by $H_{22} = v_{j_1}v_{j_2}\cdots v_{j_{h_2}}$, where $j_1, j_2, \cdots, j_{h_2} \in [n_2]$. If $(v_{j_{h_2}}, v_{j_1}) \in A(D_2)$, then $H_{22} + (v_{j_{h_2}}, v_{j_1})$ is a spanning eulerian ditrail of D_2, so D_2 is supereulerian. By (i), $D_1 \times D_2$ is supereulerian, thus, $D_1 \times D_2$ is trailable. If $(v_{j_{h_2}}, v_{j_1}) \notin A(D_2)$, we obtain a new digraph $D_{2'}$ such that $V(D_{2'}) = V(D_2)$ and $A(D_{2'}) = A(D_2) \cup (v_{j_{h_2}}, v_{j_1})$. Then $H_{22}' = H_{22} + (v_{j_{h_2}}v_{j_1})$ is a spanning closed ditrail in $D_{2'}$. Let

$$H' = (\cup_{j=1}^{n_2} H_1^{v_j}) \cup H'^{u_1}_{22} = (\cup_{j=1}^{n_2} H_1^{v_j}) \cup (H_{22}^{u_1} + ((u_1, v_{j_{h_2}}), (u_1, v_{j_1}))).$$

By Theorem 2.3, H' is a spanning closed ditrail in $D_1 \times D_2$. Let

$$H = H' - ((u_1, v_{j_{h_2}}), (u_1, v_{j_1})) = (\cup_{j=1}^{n_2} H_1^{v_j}) \cup H_{22}^{u_1}.$$

Then H is a spanning ditrail in $D_1 \times D_2$.

A digraph D is **bi-trailable** if there exist two distinct vertices $x, y \in V(D)$, such that D has both spanning (x, y)-ditrail and spanning (y, x)-ditrail. In the study of supereulerian and trailable Cartesian product of digraphs, bi-trailable digraphs seem to play a useful role.

Theorem 2.4 Let D_1 be a bi-trailable digraph and D_2 be a digraph.
(i) If D_2 is trailable, then the Cartesian product $D_1 \times D_2$ is trailable.
(ii) If D_2 is supereulerian with $|V(D_2)| \geq 2$ and $|V(D_2)|$ is even, then the Cartesian product $D_1 \times D_2$ is supereulerian.

Proof. Let $V(D_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(D_2) = \{v_1, v_2, \cdots, v_{n_2}\}$. Since D_1 is bi-trailable, we assume that for a pair of distinct vertices $x, y \in V(D_1)$, D_1 contains a spanning (x, y)-ditrail $H_{11} = u_s u_{i_1} u_{i_2} \cdots u_{i_h} u_t$ and a spanning (y, x)-ditrail $H_{12} = u_t u_{l_1} u_{l_2} \cdots u_{l_{h'}} u_s$, where $x = u_s$, $y = u_t$ and $s, t, i_1, i_2, \cdots, i_h, l_1, l_2, \cdots, l_{h'} \in [n_1]$. If L_i is a subdigraph of D_i for $i = 1,2$, then for each $u_j \in V(D_1)$ and $v_k \in V(D_2)$, we use $L_1^{v_k}$ to denote the the corresponding subdigraph in $D_1^{v_k}$ and $L_2^{u_j}$ to denote the corresponding subdigraph in $D_2^{u_j}$.

To prove (i), we present an algorithm (Algorithm A below) to find a spanning ditrail in $D_1 \times D_2$. By assumption, D_2 has a spanning ditrail H_2. Denote $H_2 = v_{j_1}v_{j_2}\cdots v_{j_{h_2}}$, with $j_1, j_2, \cdots, j_{h_2} \in [n_2]$. The algorithm has a set A to record vertices $v_j \in V(D_2)$ that the ditrail has visited the copy $D_1^{v_j}$, and will start from a vertex (u_s, v_{j_1}), travel along $H_{11}^{v_{j_1}}$ in $D_1^{v_{j_1}}$ to end at (u_t, v_{j_1}), and place v_{j_1} in A. Then in $D_2^{u_t}$, move to (u_t, v_{j_2}) and travel along $H_{12}^{v_{j_2}}$ in $D_1^{v_{j_2}}$ to end at (u_s, v_{j_2}), and place v_{j_2} in A. Inductively, at $(u_t, v_{j_{r-1}})$, if $v_{j_r} \in A$, that is, $D_1^{v_{j_r}}$ has been traversed, then in $D_2^{u_t}$, move to (u_t, v_{j_r}); if $v_{j_r} \not\in A$, that is, $D_1^{v_{j_r}}$ has not been traversed, then in $D_2^{u_t}$, move to (u_t, v_{j_r}) and travel along $H_{12}^{v_{j_r}}$ in $D_1^{v_{j_r}}$ to end at (u_s, v_{j_r}), and place v_{j_r} in A. A similar process will be done if at $(u_s, v_{j_{r-1}})$, until all vertices in $D_1 \times D_2$ are visited.

Algorithm A:

INPUT: A digraph D_1 with spanning ditrails H_{11} and H_{12} and a digraph D_2 with spanning ditrail H_2, define $H_2' = \{v_{j_1}(1), v_{j_2}(2), \cdots, v_{j_r}(p), \cdots, v_{j_{h_2}}(q)\}$. Using the notation above.

OUTPUT: A spanning ditrail H in $D_1 \times D_2$ starting from (u_s, v_{j_1}).

1. Let $H := H_{11}^{v_{j_1}}$; $A := \{v_{j_1}\}$ and $p := 2$.

2. If $p > q$, go to step 6.

3. Let H be current ditrail.

If $(u_t, v_{j_{r-1}})$ is the terminal vertex of H, go to step 4.

If $(u_s, v_{j_{r-1}})$ is the terminal vertex of H, go to step 5.

4. If $v_{j_r} \in A$ for $v_{j_r}(p) \in H_{2'}$, set $H := H + ((u_t, v_{j_{r-1}}), (u_t, v_{j_r}))$, $A := A \cup \{v_{j_r}\}$ and $p := p + 1$, go to step 2.

If $v_{j_r} \notin A$ for $v_{j_r}(p) \in H_{2'}$, set $H := (H + ((u_t, v_{j_{r-1}}), (u_t, v_{j_r}))) \cup H_{12}^{v_{j_r}}$, $A := A \cup \{v_{j_r}\}$ and $p := p + 1$, go to step 2.

5. If $v_{j_r} \in A$ for $v_{j_r}(p) \in H_{2'}$, set $H := H + ((u_s, v_{j_{r-1}}), (u_s, v_{j_r}))$, $A := A \cup \{v_{j_r}\}$ and $p := p + 1$, go to step 2.

If $v_{j_r} \notin A$ for $v_{j_r}(p) \in H_{2'}$, set $H := (H + ((u_s, v_{j_{r-1}}), (u_s, v_{j_r}))) \cup H_{11}^{v_{j_r}}$, $A := A \cup \{v_{j_r}\}$ and $p := p + 1$, go to step 2.

6. Return the ditrail H.

The finiteness of D_1 and D_2 indicates that the Algorithm will terminate. Let H be the output of Algorithm A. We are to show that H is a spanning ditrail. In fact, at each step of Algorithm A, the current H is always a ditrail. As $V(H_{11}) = V(H_{12}) = V(D_1)$, and as by Steps 1, 3, 4, 5 in Algorithm A, we note that $V(H) = \cup_{k=j_1}^{j_{h_2}} V(D_1^{v_k})$ and $\{v_{j_1}, v_{j_2}, \cdots, v_{j_{h_2}}\} = \mathrm{V}(D_2)$. By Observation 2.1 (i) and (ii), H is a spanning ditrail of $D_1 \times D_2$. This proves (i).

We will construct a spanning closed ditrail H' of $D_1 \times D_2$ to prove (ii). Recall that H_{11} and H_{12} are spanning ditrails of D_1. Let $H_2 = v_{j_1} v_{j_2} \cdots v_{j_{h_2}} v_{j_1}$ be a spanning closed ditrail in D_2. Then $H_{2'} = v_{j_1} v_{j_2} \cdots v_{j_{h_2}}$ is a spanning ditrail in D_2. By Algorithm A, and since $|V(D_2)|$ is even, H is a spanning ditrail in $D_1 \times D_2$ starting from (u_s, v_{j_1}) and ending at $(u_s, v_{j_{h_2}})$. Since $(v_{j_{h_2}}, v_{j_1}) \in A(D_2)$, it follows by the definition of Cartesian product that $((u_s, v_{j_{h_2}}), (u_s, v_{j_1})) \in A(D_1 \times D_2)$. It follows that the subdigraph $H + ((u_s, v_{j_{h_2}}), (u_s, v_{j_1}))$ is a spanning closed ditrail in $D_1 \times D_2$. This proves (ii).

2.3 Lexicographic product ofdigraphs

Sufficient conditions on $D_1$and $D_2$for the Lexicographic product $D_1[D_2]$ to be supereulerian or trailable will be investigated in this section.

Theorem 2.5 Let $D_1$and D_2be two digraphs. If $D_1$is supereulerian with $|V(D_1)| \geq 2$, then the Lexico- graphic product $D_1[D_2]$ is supereulerian.

Proof. Let$V(D_1)=\{u_1,u_2,\cdots,u_{n1}\}andV(D_2)=\{v_1,v_2,\cdots,v_{n2}\}$.If$V(D_2)=1$,then$D_1[D_2] \cong D_1$ is supereulerian. Hence we assume that $|V(D_2)| \geq 2$. As $D_1$is supereulerian, we assume that $H_1= u_{i1}\, u_{i2} \cdots u_{ih}\, u_{i1}$ is a spanning closed ditrailofD_1.

(5) By (5), $(u_{ih}, u_{i1}) \in A(D_1)$, and so by the definition of Lexicographic product of digraphs, foranyvertices$v_s,v_t \in V(D_2)$,wehavethat$((u_{ih},v_s),(u_{i1},v_t)) \in A(D_1[D_2])$.(6)

To construct a spanning closed ditrail of $D_1[D_2]$, we start with (u_{i_1}, v_1) in $D_1^{v_1}$, traveling along $H_1^{v_1}$ ending at (u_{i_h}, v_1); and then by (6), use the arc $((u_{i_h}, v_1), (u_{i_1}, v_2))$ to move to (u_{i_1}, v_2). Inductively, for some $p < n_2$, the ditrail at (u_{i_1}, v_p) in $D_1^{v_p}$, will travel along $H_1^{v_p}$ to end at (u_{i_h}, v_p). When $p = n_2$, the ditrail applies (6) again and takes the arc $((u_{i_h}, v_p), (u_{i_1}, v_1))$ to complete the ditrail, which is now a spanning closed ditrail of $D_1[D_2]$. The construction of this spanning closed ditrail of $D_1[D_2]$ can be illustrated in Algorithm B below.

Algorithm B:

INPUT: A digraph D_1 with a spanning closed ditrail $H_1 = u_{i_1} u_{i_2} \cdots u_{i_h} u_{i_1}$ and a digraph D_2 with $V(D_2) = \{v_1, v_2, \cdots, v_{n_2}\}$.

OUTPUT: A spanning closed ditrail H in $D_1[D_2]$.

1. Let $H := H_1^{v_1} - ((u_{i_h}, v_1), (u_{i_1}, v_1))$; $p := 2$.

2. If $p > n_2$, let $H := H + ((u_{i_h}, v_{n_2}), (u_{i_1}, v_1))$, go to step 5.

3. Let H be current ditrail with the terminal vertex (u_{i_h}, v_{p-1}).

4. Set $H := (H + ((u_{i_h}, v_{p-1}), (u_{i_1}, v_p))) \cup H_1^{v_p} - ((u_{i_h}, v_p), (u_{i_1}, v_p))$ and $p := p + 1$, go to step 2.
5. Return the closed ditrail H.

As in each step of Algorithm B, the current H is a ditrail starting from (u_{i_1}, v_1) in $D_1^{v_1}$, the finiteness of the digraphs implies that Algorithm B must stop. When Step 2 is executed, H becomes a closed ditrail. Since H_1 is a spanning ditrail of D_1, we have $V(H_1) = V(D_1)$. By steps 1, 3, 4, 5, $V(H) = \cup_{k=j_1}^{j_{h_2}} V(D_1^{v_k})$ and $\{v_{j_1}, v_{j_2}, \cdots, v_{j_{h_2}}\} = V(D_2)$. Thus at the end of the algorithm, we have

$$H = ((\bigcup_{j=1}^{n_2} (H_1^{v_j} - ((u_{i_h}, v_j), (u_1, v_j)))) + (\bigcup_{t=1}^{n_2-1} ((u_{i_h}, v_t), (u_{i_1}, v_{t+1})))) + ((u_{i_h}, v_{n_2}), (u_{i_1}, v_1)).$$

Therefore, H is a spanning closed ditrail of $D_1[D_2]$. This proves the theorem.

Theorem 2.6 Let D_1 and D_2 be two strong digraphs with $min\{|V(D_1)|, |V(D_2)|\} \geq 2$ and D_1 is trailable. Then the Lexicographic product $D_1[D_2]$ is supereulerian.

Proof. Let $V(D_1) = \{u_1, u_2, \cdots, u_{n_1}\}$ and $V(D_2) = \{v_1, v_2, \cdots, v_{n_2}\}$. Let H be a spanning ditrail of D_1. If H is closed, then by Theorem 2.5, $D_1[D_2]$ is supereulerian. Hence we assume that
$H_1 = u_{i_1} u_{i_2} \cdots u_{i_h}$ *is a spanning ditrail of* D_1, *where* $u_{i_s} \in V(D_1)$ *for* $s \in [h]$ *and* $u_{i_1} = u_1 \neq u_{i_h}$. (7)
For each $p \in [n_2]$, define $L_1^{v_p} = H_1^{v_p} - ((u_{i_1}, v_p), (u_{i_2}, v_p))$. Since D_1 is strong, D_1 contains a shortest (u_{i_h}, u_{i_1})-dipath $P = u_{i_{s_k}} u_{i_{s_{k-1}}} \cdots u_{i_{s_2}} u_{i_{s_1}}$, where $u_{i_{s_k}} = u_{i_h}$ and $u_{i_{s_1}} = u_{i_1}$. By the definition Lexicographic product, we observe that
for any vertices $v, v' \in V(D_2)$, *if* $(u, u') \in A(D_1)$, *then* $((u, v), (u', v')) \in A(D_1[D_2])$.(8)
We will construct a spanning closed ditrail of $D_1[D_2]$ depending on the parity of $k = |V(P)|$. Assume first that k is even, we start with (u_{i_1}, v_1) in $D_1^{v_1}$, travel along $H_1^{v_1}$ to end at (u_{i_h}, v_1); then by (8), take the dipath $P_{2'} = (u_{i_{s_k}}, v_1)(u_{i_{s_{k-1}}}, v_2)(u_{i_{s_{k-2}}}, v_1)(u_{i_{s_{k-3}}}, v_2) \cdots (u_{i_{s_2}}, v_1)(u_{i_{s_1}}, v_2)$ to reach $(u_{i_{s_1}}, v_2)$. Inductively, for some p with $2 \leq p \leq n_2$, the current ditrail will move from $(u_{i_{s_1}}, v_p)$, traversing along $H_1^{v_p}$ ending at $((u_{i_h}, v_p)$; then take the dipath

$$P_{p'} = (u_{i_{s_k}}, v_{p-1})(u_{i_{s_{k-1}}}, v_p)(u_{i_{s_{k-2}}}, v_{p-1})(u_{i_{s_{k-3}}}, v_p) \cdots (u_{i_{s_2}}, v_{p-1})(u_{i_{s_1}}, v_p) \quad (9)$$

to reach $(u_{i_{s_1}}, v_p)$. At $(u_{i_{s_k}}, v_{n_2})$, it utilizes (8) to take

$$P_{1'} = (u_{i_{s_k}}, v_{n_2})(u_{i_{s_{k-1}}}, v_1)(u_{i_{s_{k-2}}}, v_{n_2})(u_{i_{s_{k-3}}}, v_1) \cdots (u_{i_{s_2}}, v_{n_2})(u_{i_{s_1}}, v_1) \quad (10)$$

to return to $(u_{i_{s_1}}, v_1)$.
Assume next that k is odd, we start with (u_{i_1}, v_1) in $D_1^{v_1}$, travel along $H_1^{v_1}$ to end at (u_{i_h}, v_1); then by (8), take the dipath $P_{2''} = (u_{i_{s_k}}, v_1)(u_{i_{s_{k-1}}}, v_2)(u_{i_{s_{k-2}}}, v_1)(u_{i_{s_{k-3}}}, v_2) \cdots (u_{i_{s_2}}, v_2)(u_{i_{s_1}}, v_1)$ and then bypass $((u_{i_{s_1}}, v_1), (u_{i_2}, v_2))$ to reach (u_{i_2}, v_2). Inductively, for some p with $2 \leq p \leq n_2$, the current ditrail will move from (u_{i_2}, v_p), travel along $L_1^{v_p} = H_1^{v_p} - ((u_{i_1}, v_p), (u_{i_2}, v_p))$ to get to (u_{i_h}, v_p); then take the dipath

$$P_{p''} = (u_{i_{s_k}}, v_{p-1})(u_{i_{s_{k-1}}}, v_p)(u_{i_{s_{k-2}}}, v_{p-1})(u_{i_{s_{k-3}}}, v_p) \cdots (u_{i_{s_2}}, v_p)(u_{i_{s_1}}, v_{p-1})(u_{i_2}, v_p) \quad (11)$$

to reach (u_{i_2}, v_p). At (u_{i_2}, v_{n_2}), the current ditrail will travel along $L_1^{v_{n_2}} = H_1^{v_{n_2}} - ((u_{i_1}, v_{n_2}), (u_{i_2}, v_{n_2}))$ to get to (u_{i_h}, v_{n_2}), then following

$$P_{1''} = (u_{i_{s_k}}, v_{n_2})(u_{i_{s_{k-1}}}, v_1)(u_{i_{s_{k-2}}}, v_{n_2})(u_{i_{s_{k-3}}}, v_1) \cdots (u_{i_{s_2}}, v_1)(u_{i_{s_1}}, v_{n_2}) \quad (12)$$

to arrive at (u_{i_1}, v_{n_2}). Since $D_2^{u_{i_1}} \cong D_2$ is strong, $D_2^{u_{i_1}}$ has a $((u_{i_1}, v_{n_2}), (u_{i_1}, v_1))$-dipath $P_2^{u_{i_1}}$. Then, from (u_{i_1}, v_{n_2}), it goes through $P_2^{u_{i_1}}$ to return to (u_{i_1}, v_1).
With the definitions of the related dipaths in (9), (10), (11), (12), the construction of this spanning closed ditrail of $D_1[D_2]$ can be illustrated in Algorithm C below.

Algorithm C:
INPUT: A strong digraph D_1 with a spanning ditrail and a strong digraph D_2 .
OUTPUT: A spanning closed ditrail H in $D_1[D_2]$.
1. Let $H := H_1^{v_1}$ and $p := 2$.
2. If $p > n_2$, and
if k is even, let $H := H \cup P_{1'}$, go to step 5;
if k is odd, let $H := H \cup P_{1''} \cup P_2^{u_{i_1}}$, go to step 5.

3. Let H be current ditrail with the terminal vertex (u_{i_h}, v_{p-1}).
4. If k is even, let $H := H \cup P_{p'} \cup H_1^{v_p}$ and $p := p + 1$, go to step 2.
If k is odd, let $H := H \cup P_{p''} \cup H_{1'}^{v_p}$ and $p := p + 1$, go to step 2.
5. Return the closed ditrail H.
As in each step of Algorithm C, the current H is a ditrail starting from (u_{i_1}, v_1) in $D_1^{v_1}$, the finiteness of the digraphs implies that Algorithm C must stop. When Step 2 is executed, H becomes a closed ditrail. Since H_1 is a spanning ditrail of D_1, we have $V(H_1) = V(D_1)$. By steps 1, 3, 4, 5, $V(H) = \cup_{k=j_1}^{j_{h_2}} V(D_1^{v_k})$ and $\{v_{j_1}, v_{j_2}, \cdots, v_{j_{h_2}}\} = V(D_2)$, thus, by Observation 2.1 (i) and (ii), H is a spanning closed ditrail of $D_1[D_2]$.
Since a bi-trailable digraph is strong, by Theorem 2.6 the following corollary holds.
Corollary 2.2 Let D_1 and D_2 be two digraphs with $\min\{|V(D_1)|, |V(D_2)|\} \geq 2$.

- If D_1 is a bi-trailable digraph and D_2 is a strong digraph, then the Lexicographic product $D_1[D_2]$ is supereulerian.
- If D_1 is trailable and strong, then the Lexicographic product $D_1[D_2]$ is trailable.

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