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Common Fixed Point Theorem in Menger Space through weak Compatibility

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Abstract: *In the present paper, a common fixed point theorem for five self mappings has been proved under more general t-norm (H -type norm) in Menger space through weak compatibility. A corollary is also derived from the obtained result. The theorem is supported by providing a suitable example.*
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I. Introduction

Fixed point theory in Menger space can be considered as a part of Probabilistic Analysis, which is a very dynamical area of mathematical research. Menger [1] introduced the notion of Menger spaces as a generalization of core notion of metric spaces. It is observed by many authors that contraction condition in metric space may be exactly translated into PM-space endowed with min norms. Sehgal and Bharucha-Reid [2] obtained a generalization of Banach Contration Principle on a complete Menger space which is a milestone in developing fixed-point theorems in Menger space and initiated the study of fixed points in PM-spaces. Further, Schweizer-Sklar [3] expanded the study of these spaces.

Mishra [4] introduced the notion of compatible mappings in PM-space. Jungck [5] enlarged this concept of compatible maps. Sessa [6] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly compatible commuting maps in Metric spaces.

In the present paper, using the idea of compatibility, we have proved a common fixed point theorem for five self mappings in Complete Menger space and an example is also given to illustrate our proved theorem. Also we have deduced a corollary from main theorem.

II. Preliminary Notes

In this section, we recall some definitions and known results in Menger space.

Definition 2.1 A distribution function is a function $F:[-\infty,\infty]\to[0,1]$ which is left continuous on R, non-decreasing and $F(-\infty)=0, F(\infty)=1$.

Let $\Delta=\{F: F\ is\ distribution\ function\}$ and $H\in\Delta$ (also known as Heaviside function) defined by

$$H(y)=\begin{cases}0 & if\ y\le 0\\ 1 & if\ y>0\end{cases}$$

Definition 2.2 A triangular norm (shortly t-norm) is a binary operation on unit interval [0,1] such that $\forall p,q,r,s\in[0,1]$ the following conditions are satisfied:

- $t(a,1)=a;$
- $t(a,b)=t(b,a);$
- $t(a,b)\le t(a,b)\ whenever\ a\le r\ and\ b\le s;$
- $t\big(a,t(b,r)\big)=t\big(t(a,b),t(r)\big).$

Example 2.3 The following are the four basic t -norms:

i. The minimum t t -norm : $t_m(a,b)=min\{a,b\}$
ii. The product t-norm: $t_p\ (a,b)$ =a.b
iii. The Lukasiewicz t-norm: $t_L(a,b)=max\{a+b-1,0\}$
iv. The weakest t -norm, the drastic product: $t_D(a,b)=\begin{cases}min\{a,b\} & if\ max\{a,b\}=1\\ 0 & otherwise\end{cases}$.

In respect of the above mentioned norms, we have the following ordering:
$t_D<t_L<t_p<t_m$.

Definition 2.4:- The ordered pair (Y,G) is called a probabilistic metric space (PM-space) if Y is a non- empty set G is a probabilistic distance function satisfying the following condition: $\forall a,b,c\in Y\ and\ r,s>0,$

i. $G_{a,b}(r)=0\Leftrightarrow a=b;$

ii. $G_{a,b}(0) = 0$;
iii. $G_{a,b}(r) = G_{b,a}(r), \forall r > 0$;
iv. $G_{a,c}(r) = 1, G_{c,b}(s) = 1 \Rightarrow G_{a,b}(r+s) = 1$.

With the following additional condition, the ordered triplet (Y, G, t) is called Menger space if (Y, G) is a PM-space, t is a t -norm and$\forall a, b, c \in Y$ *and* $r, s > 0$,

v. $G_{a,b}(r+s) \geq t\left(G_{a,c}(r), G_{c,b}(s)\right)$.

This is known as Menger's Inequality (Schweizer and Sklar [3]).

Preposition1. Let (Y, d) be a metric space. Then the metric d induces a distribution function G defined by $G_{u,v}(t) = H(t - d(u,v))\ \forall u, v \in Y$ *and* $t > 0$, (Sehgal and Bharucha-Reid [2]).

If t -norm is given by, $t(a,b) = min\{a,b\}\ \forall a, b \in [0,1]$ then (Y, G, t) is a Menger space. Further, (Y, G, t) is complete Menger space if (Y, d) is complete.

Definition 2.5 Let (Y, G, t) be a Menger space and t be a continuous t -norm.

I. A sequence $\{y_n\}$ in (Y, G, t) is said to be convergent to a point $y \in Y$ if for every $\varepsilon > o$ *and* $\lambda > 0$, there exists an integer N=N(ε, λ) such that $y_n \in U_u(\varepsilon, \lambda) \forall n \geq N$ or equivalently $G_{y_n,y}(\varepsilon) > 1 - \lambda\ \forall n \geq N$; where $U_u(\varepsilon, \lambda) = \{v \in Y; G_{u,v}(\varepsilon) > 1 - \lambda\}$ is (ε, λ) neighborhood of u$\in Y$ and $\lambda \in (0,1)$.

II. A sequence $\{y_n\}$ in (Y, G, t) is said to be Cauchy sequence if for every $\varepsilon > o$ *and* $\lambda > 0$, there exists an integer N=N(ε, λ) such that $G_{y_n,y_m}(\varepsilon) > 1 - \lambda\ \forall n, m \geq N$.

III. A Menger space (Y, G, t) with continuous t -norm t is said to be complete if every Cauchy sequence in Y converges to a point in Y, (Singh et al [7]).

Definition 2.6 In a Menger space (Y, G, t) two self mappings F_1 *and* F_2 are said to be weakly compatible or coincidentally commuting if they commute at their coincidence points, i.e. if $F_1(y) = F_2(y)$ *for some* $y \in Y$ then $F_1F_2(y) = F_2F_1(y)$, (Singh & Jain [8]).

Definition 2.7 Two self mappings F_1 *and* F_2 in a Menger space (Y, G, t) are called compatible if $G_{F_1F_2(y_n),F_2F_1(y_n)}(t) \to 1\ \forall\ t > 0$, whenever $\{y_n\}$ is a sequence in $Y: F_1(y_n), F_2(y_n) \to y$, for some $y \in Y$, *as* $n \to \infty$ (Mishra [4]).

Preposition2. In a Menger space (Y, G, t) two self mappings F_1 *and* F_2 are compatible then they are weakly compatible. But converse of the above result is not true (Singh & Jain [8]).

Illustration of converse part with example:

Example 1 Let (Y, d) be a metric space $Y = [0,3]$ and (Y, G, t) be the induced Menger space with$G_{a,b}(t) = H(t - d(a,b)), \forall\ a, b \in Y$ *and* $\forall t > 0$. Define self maps F_1 *and* F_2 as follows:

$F_1(y) = \begin{cases} 3 - y & if 0 \leq y < 2, \\ 3 & if 2 \leq y \leq 3, \end{cases}$ and $F_2(y) = \begin{cases} y - 1 & if 0 \leq y < 2, \\ 3 & if 2 \leq y \leq 3, \end{cases}$

Take $y_m = 2 - {}^{1}/_{m}$. Now,

$G_{F_1(y_m),1}(t) = H\left(t - \left({}^{1}/_{m}\right)\right)$; therefore $\lim_{m\to\infty} G_{F_1(y_m),1}(t) = H(t) = 1$.

Hence $F_1(y_m) \to 1$ *as* $m \to \infty$. Similarly, $F_2(y_m) \to 1$ *as* $m \to \infty$. Also

$G_{F_1F_2(y_m),F_2F_1(y_m)}(t) = H(t - (2)), \lim_{m\to\infty} G_{F_1F_2(y_m),F_2F_1(y_m)}(t) = H(t-2) \neq 1, \forall t > 0$.

Hence, the pair (F_1, F_2) is not compatible. Also set of coincidence points of F_1 *and* F_2 is $[2,3]$. Now for any $y \in [2,3], F_1(y) = F_2(y) = 3$, and $F_1F_2(y) = F_1(3) = 3 = F_2(3) = F_2F_1(y)$. Thus$F_1$ *and* F_2 are weakly compatible but not compatible.

Preposition3. In a Menger space (Y, G, t), if $t(a,a) \geq a\ \forall a \in [0,1]$, then $t(a,b) = min\{a,b\}\ \forall a, b \in [0,1]$, (Singh & Jain [8]).

Lemma 1 (Singh & Pant [9]) Let $\{y_n\}$ be a sequence in a Menger space (Y, G, t) with continuous t-norm and $t(a,a) \geq a$. Suppose $\forall a \in [0,1], \exists c \in (0,1): \forall t > 0$ *and* $n \in N$,

$G_{y_n,y_{n+1}}(ct) \geq G_{y_{n-1},y_n}(t)$. Then $\{y_n\}$ is a Cauchy sequence in Y, (Singh & Pant [9]).

Lemma 2 Let (Y, G, t) be a Menger space. If $\exists c \in (0,1)$ such that for $a, b \in Y, G_{a,b}(ct) \geq G_{a,b}(t)$. Then $a = b$, (Singh & Jain [8]).

III. Main Result

Theorem 3.1 Let F_1, F_2, F_3, F_4 *and* F_5 are self mappings on a complete Menger space $(Y, G, t)$with$t(a,a) \geq a\ \forall a \in [0,1]$, satisfying:

(I) $F_4 \subseteq F_1F_2, F_5 \subseteq F_3$;
(II) $F_1F_2 = F_2F_1, F_4F_2 = F_2F_4, F_3F_2 = F_2F_3, F_1F_5 = F_5F_1$;
(III) Either F_4 or F_3 is continuous;
(IV) (F_4, F_3) is compatible & (F_5, F_1F_2) is weakly compatible;
(V) $\exists c \in (0,1)$:

$$G_{F_4(a),F_5(b)}(cy) \geq min\{G_{F_3(a),F_4(a)}(y);\ G_{F_1F_2(b),F_5(b)}(y);\ G_{F_1F_2(b),F_4(a)}(\alpha y);\ G_{F_3(a),F_5(b)}((2-\alpha)y);\ G_{F_3(a),F_1F_2(b)}(y)\} \ \forall\ a, b \in Y;\ \alpha \in (0,2)\ \&\ y > 0.$$

Proof: Let $y_0 \in Y$. From condition (I) $\exists y_1, y_2 \in Y$:
$F_4(y_0) = F_1F_2(y_1) = z_0$ and
$F_5(y_1) = F_3(y_2) = z_1$.
Inductively we can construct sequences $\{y_n\}$ *and* $\{z_n\}$ in Y such that
$F_4(y_{2n}) = F_1F_2(y_{2n+1}) = z_{2n}$ and
$F_5(y_{2n+1}) = F_3(y_{2n+2}) = z_{2n+1}$ for n=0,1,2_ _ _ _ _.
Step1. Putting $a = y_{2n}, b = y_{2n+1}, \alpha = 1 - q$ with $q \in (0,1)$ in (V), we get

$$\boldsymbol{G_{F_4(y_{2n}),F_5(y_{2n+1})}(cy) \geq min\{G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);\ G_{F_1F_2(y_{2n+1}),F_4(y_{2n})}((1-q)y);\ G_{F_3(y_{2n}),F_5(y_{2n+1})}((1+q)y);\ G_{F_3(y_{2n}),F_1F_2(y_{2n+1})}(y)\}} \quad \textbf{(1)}$$

$$\Rightarrow G_{z_{2n},z_{2n+1}}(cy) \geq min\{G_{z_{2n-1},z_{2n}}(y);\ G_{z_{2n},z_{2n+1}}(y);\ 1;\ G_{z_{2n-1},z_{2n+1}}((1+q)y);\ G_{z_{2n-1},z_{2n}}(y)\}$$
$$\geq min\{G_{z_{2n-1},z_{2n}}(y);\ G_{z_{2n},z_{2n+1}}(y);\ G_{z_{2n-1},z_{2n}}(y);\ G_{z_{2n},z_{2n+1}}(qy)\}$$
$$\geq min\{G_{z_{2n-1},z_{2n}}(y);\ G_{z_{2n},z_{2n+1}}(y);\ G_{z_{2n},z_{2n+1}}(qy)\}$$

As t -norm t is continuous, letting $q \to 1$, we get;

$$G_{z_{2n},z_{2n+1}}(cy) \geq min\{G_{z_{2n-1},z_{2n}}(y);\ G_{z_{2n},z_{2n+1}}(y); G_{z_{2n},z_{2n+1}}(y)\}$$
$$= min\{G_{z_{2n-1},z_{2n}}(y);\ G_{z_{2n},z_{2n+1}}(y)\}$$

Hence, $G_{z_{2n},z_{2n+1}}(cy) \geq\ min\{G_{z_{2n-1},z_{2n}}(y);\ G_{z_{2n},z_{2n+1}}(y)\}$
Similarly, $G_{z_{2n+1},z_{2n+2}}(cy) \geq\ min\{G_{z_{2n},z_{2n+1}}(y);\ G_{z_{2n+1},z_{2n+2}}(y)\}$. Therefore, for all n even or odd we have: $G_{z_n,z_{n+1}}(cy) \geq\ min\{G_{z_{n-1},z_n}(y);\ G_{z_n,z_{n+1}}(y)\}$
Consequently, $G_{z_n,z_{n+1}}(y) \geq\ min\{G_{z_{n-1},z_n}(c^{-1}y);\ G_{z_n,z_{n+1}}(c^{-1}y)\}$
By repeated application of above inequality, we get:
$G_{z_n,z_{n+1}}(y) \geq\ min\{G_{z_{n-1},z_n}(c^{-1}y);\ G_{z_n,z_{n+1}}(c^{-m}y)\}$
Since $G_{z_n,z_{n+1}}(c^{-m}y) \to 1\ as\ m \to \infty$, it follows that
$G_{z_n,z_{n+1}}(cy) \geq G_{z_{n-1},z_n}(y)\ \forall n \in \mathbb{N}\ \&\ \forall\ y > 0.$
Therefore, by Lemma 1, $\{z_n\}$ is a Cauchy sequence in Y, Which is complete. Hence $\{z_n\} \to z \in Y$. Also its sub-sequences converges as follows:
$\{F_5(y_{2n+1})\} \to z\ and\ \{F_1F_2(y_{2n+1})\} \to z$
$\{F_4(y_{2n})\} \to z\ and\ \{F_3(y_{2n})\} \to z$
Case I. F_3 is continuous.
As F_3 is continuous, $(F_3)^2(y_{2n}) \to F_3z$ and $(F_3)F_4(y_{2n}) \to F_3(z)$. As (F_4, F_3) is compatible, we have $F_4(F_3)(y_{2n}) \to F_3(z)$.
Step2. Putting $a = F_3(y_{2n}), b = y_{2n+1}$ with $\alpha = 1$ in condition (V), we get:

$$\boldsymbol{G_{F_4(F_3(y_{2n})),F_5(y_{2n+1})}(cy) \geq min\left\{\begin{matrix} G_{F_3(F_3(y_{2n})),F_4(F_3(y_{2n}))}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);\ G_{F_1F_2(y_{2n+1}),F_4(F_3(y_{2n}))}(y); \\ G_{F_3(F_3(y_{2n})),F_5(y_{2n+1})}(y);\ G_{F_3(F_3(y_{2n})),F_1F_2(y_{2n+1})}(y) \end{matrix}\right\}} \quad \textbf{(2)}$$

Letting n→ ∞ we get
$G_{F_3(z),z}(cy) \geq min\{G_{F_3(z),F_3(z)}(y);\ G_{z,z}(y);\ G_{z,F_3(z)}(y);\ G_{F_3(z),z}(y);\ G_{F_3(z),z}(y)\}$,
i.e. $G_{F_3(z),z}(cy) \geq G_{F_3(z),z}(y)$. Therefore, by Lemma 2, we get
$F_3(z) = z.$
Step3. Putting $a = z, b =\ y_{2n+1}$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(z),F_5(y_{2n+1})}(cy) \geq min \begin{Bmatrix} G_{F_3(z),F_4(z)}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);\ G_{F_1F_2(y_{2n+1}),F_4(z)}(y); \\ G_{F_3(z),F_5(y_{2n+1})}(y);\ G_{F_3(z),F_1F_2(y_{2n+1})}(y) \end{Bmatrix} \quad (3)$$

Letting n→ ∞ we get

$G_{F_4(z),z}(cy) \geq min\{G_{z,F_4(z)}(y);\ G_{z,z}(y);\ G_{z,F_4(z)}(y);\ G_{z,z}(y);\ G_{z,z}(y)\}$, i.e.

$G_{F_4(z),z}(cy) \geq G_{F_4(z),z}(y)$, which gives $F_4(z) = z$. Therefore $F_3(z) = F_4(z) = z$.

Step4. Putting $a = F_2(z), b = y_{2n+1}$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(F_2(z)),F_5(y_{2n+1})}(cy) \geq min \begin{Bmatrix} G_{F_3(F_2(z)),F_4(F_2(z))}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);\ G_{F_1F_2(y_{2n+1}),F_4(F_2(z))}(y); \\ G_{F_3(F_2(z)),F_5(y_{2n+1})}(y);\ G_{F_3(a),F_1F_2(b)}(y) \end{Bmatrix} \quad (4)$$

As

$F_2F_4 = F_4F_2,\ F_2F_3 = F_3F_2,$ *so we have* $F_3(F_2(z)) = F_2(F_3(z)) = F_2(z)$ *and* $F_4(F_2(z)) = F_2(F_4(z)) = F_2(z)$.

Letting n→ ∞ we get:

$G_{F_2(z),F_5(b)}(cy) \geq min\{G_{F_2(z),F_2(z)}(y);\ G_{z,z}(y);\ G_{z,F_2(z)}(y);\ G_{F_2(z),z}(y);\ G_{F_2(z),z}(y)\}$, i.e.

$G_{F_2(z),F_5(b)}(cy) \geq G_{F_2(z),F_5(b)}(y);$ which gives $F_2(z) = z$. Therefore,

$F_3(z) = F_2(z) = F_4(z) = z.$

Step5. As $F_4(Y) \subseteq F_1F_2(Y), \exists\ u \in Y : z = F_4(z) = F_1F_2(u)$.

Putting $a = y_{2n}, b = u$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(y_{2n}),F_5(u)}(cy) \geq min\{G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{F_1F_2(u),F_5(u)}(y);\ G_{F_1F_2(u),F_4(y_{2n})}(y);\ G_{F_3(y_{2n}),F_5(u)}(y);\ G_{F_3(y_{2n}),F_1F_2(u)}(y)\} \quad (5)$$

Letting n→ ∞ and we get:

$G_{z,F_5(u)}(cy) \geq min\{G_{z,z}(y);\ G_{z,F_5(u)}(y);\ G_{z,z}(y);\ G_{z,F_5(u)}(y);\ G_{z,z}(y)\};$ i.e.

$G_{z,F_5(u)}(cy) \geq G_{z,F_5(u)}(y)$. Therefore by Lemma 2, $F_5(u) = z$. Hence $F_1F_2(u) = z = F_5(u)$. As (F_5, F_1F_2) is weakly compatible, We have $F_1F_2F_5(u) = F_5F_1F_2(u)$. Thus, $F_1F_2(z) = F_5(u)$.

Step6. Putting $a = y_{2n}, b = z$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(y_{2n}),F_5(z)}(cy) \geq min\{G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{F_1F_2(z),F_5(z)}(y);\ G_{F_1F_2(z),F_4(y_{2n})}(y);\ G_{F_3(y_{2n}),F_5(z)}(y);\ G_{F_3(y_{2n}),F_1F_2(z)}(y)\} \quad (6)$$

Letting n→ ∞ and using equation we get:

$G_{z,F_5(z)}(cy) \geq min\{G_{z,z}(y);\ G_{F_5(z),F_5(z)}(y);\ G_{F_5(z),z}(y);\ G_{z,F_5(z)}(y);\ G_{z,F_5(z)}(y)\}$ i.e.

$G_{z,F_5(z)}(cy) \geq G_{z,F_5(z)}(y)$. Hence, $F_5(z) = z$.

Step7. Putting $a = y_{2n}, b = F_1(z)$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(y_{2n}),F_5(F_1(z))}(cy) \geq min \begin{Bmatrix} G_{F_3(y_{2n}),F_4(y_{2n})}(y);\ G_{F_1F_2(F_1(z)),F_5(F_1(z))}(y);\ G_{F_1F_2(F_1(z)),F_4(y_{2n})}(y); \\ G_{F_3(y_{2n}),F_5(F_1(z))}(y);\ G_{F_3(y_{2n}),F_1F_2(F_1(z))}(y) \end{Bmatrix} \quad (7)$$

As $F_1F_2 = F_2F_1$ *and* $F_5F_1 = F_1F_5$ We have $F_5F_1(z) = F_1F_5(z) = F_1(z)$.

Letting n→ ∞, we get:

$G_{z,F_1(z)}(cy) \geq min\{G_{z,z}(y);\ G_{F_1(z),F_1(z)}(y);\ G_{F_1(z),z}(y);\ G_{z,F_1(z)}(y);\ G_{z,F_1(z)}(y)\}$ i.e.

$G_{z,F_1(z)}(cy) \geq G_{z,F_1(z)}(y)$. Therefore, by Lemma 2, $F_1(z) = z$. Hence,

$F_1(z) = F_2(z) = F_3(z) = F_4(z) = F_5(z) = z$.

Thus we obtain that z the common fixed point of the five maps in this case.

Case II. F_4 is continuous.

As F_4 is continuous, $(F_4)^2(y_{2n}) \to F_4(z)$ *and* $F_4(F_3(z)) \to F_4(z)$. As (F_4, F_3) is compatible, we have $F_3(F_4(z)) \to F_4(z)$.

Step8. Putting $a = F_4(y_{2n}), b = y_{2n+1}$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(F_4(y_{2n})),F_5(y_{2n+1})}(cy) \geq min\left\{\begin{matrix} G_{F_3(F_4(y_{2n})),F_4(F_4(y_{2n}))}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);\ G_{F_1F_2(y_{2n+1}),F_4(F_4(y_{2n}))}(y); \\ G_{F_3(a),F_5(y_{2n+1})}(y);\ G_{F_3(F_4(y_{2n})),F_1F_2(y_{2n+1})}(y) \end{matrix}\right\} \quad (8)$$

Letting n→ ∞, we get:

$G_{F_4(z),z}(cy) \geq min\{G_{F_4(z),F_4(z)}(y);\ G_{z,z}(y);\ G_{z,F_4(z)}(y);\ G_{F_4(z),z}(y);\ G_{F_4(z),z}(y)\}$; i.e.

$G_{F_4(z),z}(cy) \geq G_{F_4(z),z}(y)$ which gives $F_4(z) = z$.

Now steps 5-7 gives us $F_1(z) = F_4(z) = F_5(z) = F_1F_2(z) = z$.

Step9. As $F_5(Y) \subseteq F_3(z)\ \exists\ w \in Y : z = F_5(z) = F_3(w)$.

Putting $a = w, b = y_{2n+1}$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(w),F_5(y_{2n+1})}(cy) \geq min\left\{\begin{matrix} G_{F_3(w),F_4(w)}(y);\ G_{F_1F_2(y_{2n+1}),F_5(y_{2n+1})}(y);\ G_{F_1F_2(y_{2n+1}),F_4(w)}(y); \\ G_{F_3(w),F_5(y_{2n+1})}(y);\ G_{F_3(w),F_1F_2(y_{2n+1})}(y) \end{matrix}\right\} \quad (9)$$

Letting n→ ∞, we get:

$G_{F_4(w),z}(cy) \geq min\{G_{z,F_4(w)}(y);\ G_{z,z}(y);\ G_{z,F_4(w)}(y);\ G_{z,z}(y);\ G_{z,z}(y)\}$; i.e.

$G_{F_4(w),z}(cy) \geq G_{F_4(w),z}(y)$, which gives $F_4(w) = z = F_3(w)$.

As $(F_4(w), F_3(w))$ is weakly compatible, we have $F_4(z) = F_3(z)$. Also $F_1F_2 = F_2F_1, F_1(z) = z$.

So, $z = F_1F_2(z) = F_2F_1(z) = F_2(z)$.

Hence, $F_1(z) = F_2(z) = F_3(z) = F_4(z) = F_5(z) = z$, and we obtain that z is the common fixed point of the five maps in this case also.

Uniqueness: Let v be another common fixed point of F_1, F_2, F_3, F_4 and F_5; then $F_1(v) = F_2(v) = F_3(v) = F_4(v) = F_5(v) = v$. Putting $a = z, b = v$ with $\alpha = 1$ in condition (V), we get:

$$G_{F_4(z),F_5(v)}(cy) \geq min\{G_{F_3(z),F_4(z)}(y);\ G_{F_1F_2(v),F_5(v)}(y);\ G_{F_1F_2(v),F_4(z)}(y);\ G_{F_3(z),F_5(v)}(y);\ G_{F_3(z),F_1F_2(v)}(y)\} \quad (10)$$

$G_{z,v}(cy) \geq\ min\{G_{z,z}(y);\ G_{v,v}(y);\ G_{v,z}(y);\ G_{z,v}(y);\ G_{z,v}(y)\}$; i.e.

$G_{z,v}(cy) \geq G_{z,v}(y)$; which gives $z = v$. Therefore, z is unique common fixed point of F_1, F_2, F_3, F_4 and F_5.

If we take $F_2 = I$, the identity map on Y in theorem 3.1, then we get:

Corollary 3.2 Let $F_1, F_3, F_4\ and\ F_5$ are self maps on a complete Menger space (Y, G, t) with $t(a, a) \geq a\ \forall\ a \in [0,1]$, satisfying:

(I) $F_4(Y) \subseteq F_1(Y), F_5(Y) \subseteq F_3(Y)$.

(II) $F_1F_5 = F_5F_1$.

(III) Either F_4 or F_3 is continuous.

(IV) (F_4, F_3) is compatible and (F_5, F_1) is weakly compatible.

(V) There exists $c \in (0,1)$:

$$G_{F_4(a),F_5(b)}(cy) \geq min\{G_{F_1(a),F_4(a)}(y);\ G_{F_3(b),F_5(b)}(y);\ G_{F_3(b),F_4(a)}(\alpha y);\ G_{F_1(a),F_5(b)}((2 - \alpha)y);\ G_{F_1(a),F_3(b)}(y)\}\ \forall\ a, b\ \in Y, \alpha \in (0,2)\ and\ y > 0.$$

Then $F_1, F_3, F_4\ and\ F_5$ have a unique common fixed point in Y.

Now we provide an example to illustrate our proved theorem 3.1:

Example 3.3 Let $Y = [0,1]$ with the metric d defined by $d(a, b) = |a - b|$ and define $G_{a,b}(t) = H(t - d(a, b))\ \forall a, b \in Y, t > 0$. Clearly (Y, G, t) is a complete Menger space where t-norm t is defined by $t(a, b) = min\{a, b\}\ \forall a, b \in [0,1]$. Let F_1, F_2, F_3, F_4 and F_5 be maps from Y into itself defined as $F_1(y) = y, F_2(y) = \frac{y}{3}, F_3(y) = \frac{y}{7}, F_4(y) = 0, F_5(y) = \frac{y}{9}\ \ \forall y \in Y$. Then

$$F_4(Y) = \{0\} \subseteq \left[0, \frac{1}{3}\right] = F_1F_2(Y)\ and\ F_5(Y) = \left[0, \frac{1}{9}\right] \subseteq \left[0, \frac{1}{7}\right] = F_3(Y).$$

Clearly $F_1F_2 = F_2F_1, F_4F_2 = F_2F_4, F_3F_2 = F_2F_3, F_1F_5 = F_5F_1$ and F_4, F_3 are continuous. If we take $c = \frac{1}{3}\ and\ y = 1$, we see that the condition (V) of the main Theorem is also satisfied. Moreover the maps $F_4\ and\ F_3$ are compatible if $\lim_{n\to\infty} y_n = 0$, where $\{y_n\}$ is a sequence in Y such that $\lim_{n\to\infty} F_4(y_n) = 0 = \lim_{n\to\infty} F_3(y_n)\ for\ 0 \in Y$. The maps F_5 and F_1F_2 are weakly compatible at 0. Thus all conditions of the main Theorem are satisfied and 0 is the unique common fixed point of F_1, F_2, F_3, F_4 and F_5.

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