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Fixed Point Theorem in Fuzzy Metric Space Using (CLRg) Property

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ABSTRACT: *The object of this paper is to establish a common fixed point theorem for semi-compatible pair of self maps by using CLRg Property in fuzzy metric space.*
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I. INTRODUCTION

Zadeh's [1] introduced the fuzzy set theory in 1965. Zadeh's [1] introduction of the notion of fuzzy set laid the foundation of fuzzy mathematics . Sessa [2] has introduced the concept of weakly commuting and Jungck [3] initiated the concept of compatibility. In , 1988 , Jungck and Rhoades [4] introduced the notion of weakly compatible . The concept of fuzzy metric space introduced by kramosil and Mishlek [5] and modified by George and Veramani [6]. In 2009, M. Abbas et. al. [7] introduced the notion of common property E.A. B.Singh et. al. [8] introduced the notion of semi compatible maps in fuzzy metric space . Recently in 2011 , Sintunavarat and Kuman [9] introduced the concept of common limit in the range property . Chouhan et.al. [10] utilize the notion of common limit range property to prove fixed point theorems for weakly compatible mapping in fuzzy metric space .

II. PRELIMINARIES

Definition 2.1 [11] Let X be any set . A Fuzzy set A in X is a function with domain X and Values in [0,1].

Definition 2.2[6] A Binary operation * : $[0,1] \times [0,1] \to [0,1]$ is called a continuous t-norms if an topological monoid with unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$, for all a,b,c,d in [0,1].

Examples of t – norms are $a*b = ab$ and $a*b = \min\{a,b\}$.

Definition 2.3[6] The triplet (X,M, *) is said to be a Fuzzy metric space if , X is an arbitrary set , * is a continuous t- norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions; for all x,y,z in X and s,t > 0,

(i) $M(x,y,0) = 0$, $M(x,y,t)>0$,
(ii) $M(x,y,t) = 1$,for all t > 0 if and only if x=y,
(iii) $M(x,y,t) = M(y,x,t)$,
(iv) $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$,
(v) $M(x,y,t) : [0,\infty) \to [0,1]$ is left continuous.

Example 2.1 [6] Let (X,d) be a metric space . Define $a*b = \min\{a,b\}$ and $M(x,y,t) = t / t + d(x,y)$ for all $x,y \in X$ and all t > 0 . Then (X,M, *) is a fuzzy metric space . It is called the fuzzy metric space induced by the metric d.

Definition 2.4 [6] A sequence $\{x_n\}$ in a fuzzy metric space (X,M,*) is called a Cauchy **Se**quence if , $\lim_{n\to\infty} M(X_{n+p}, X_n, t) = 1$ for every t.>0 and for each p>0.

A fuzzy metric space(X, M,*) is Complete if ,every Cauchy sequence in X converge to X.

Definition 2.5[6] A sequence $\{X_n\}$ in a fuzzy metric space (X,M,*) is said to be Convergent to x in X if , $\lim_{n\to\infty} M(X_n, X, t) = 1$, for each t>0.

Definition 2.6 [12] Two self mappings P and Q of a fuzzy metric space (X,M,*) are said to be Compatible , if $\lim_{n\to\infty} M(PQx_n, QPx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$, for some z in X .

Definition 2.7 [13] Self maps A and S of a Fuzzy metric space (X,M,*) are said to be Weakly Compatible if they commute at their coincidence points,

if, AP=SP for some $p \in X$ then ASp=SAp.

Lemma 2.1 [8] Let $\{y_n\}$ is a sequence in an FM- space . If there exists a positive number k<1 such that $M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t)$, $t>0$, $n \in N$, then $\{y_n\}$ is a Cauchy sequence in X .

Lemma 2.2 [8] If for two points x, y in X and a positive number k < 1 $M(x,y,kt) \geq M(x,y,t)$, then x = y.

Lemma 2.3 [14] For all $x,y \in X$, $M(x,y,.)$ is a non – decreasing function.

Definition 2.8 [8] A pair (A,S) of self maps of a fuzzy metric space (X,M,*) is said to be Semi compatible if $\lim_{n\to\infty} ASx_n = Sx$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$, for some $x \in X$.

It follows that (A,S) is semi compatible and Ay = Sy then ASy = SAy

Example 2.2 Let X = [0,1] and (X,M, t) be the induced fuzzy metric space with $M(x,y,t) = t / t + |x-y|$. Define self maps P and Q on X as follows :

$$Px = \begin{cases} 2, & \text{if } \theta \leq x \leq 1 \\ x/2, & \text{if } 1< x\leq 2 \end{cases} \quad \text{and} \quad Qx = \begin{cases} 2, & \text{if } x=1 \\ x+3/5, & \text{if } 1 < x\leq 2 \end{cases}$$

And $x_n = 2 - 1/2^n$. Then we have P (1) = Q(1)= 2 and S(2) = A(2) = 1.

PQ(1) =QP(1) =1 and PQ(2) = QP(2) = 2 . Hence $Px_n \to 1$ and $Qx_n \to 1$ and $QPx_n \to 1$, as $n\to\infty$.

Now ,

$\lim_{n\to\infty} M(PQx_n, Qy, t) = M(2,2,t) = 1$

$\lim_{n\to\infty} M(PQx_n, QPx_n,t) = M(2,1,t) = t / 1+t < 1$.

Hence (P,Q) is semi compatible but not compatible.

Definition 2.9 [9] A pair of self mapping P and Q of a fuzzy metric space (X,M,*) is said to satisfy the (CLRg) property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = Qu$, for some $u \in X$.

Definition 2.10 [9] Two pairs (A,S) and (B,T) of self mappings of a fuzzy metric Space (X,M, *) are said to satisfy the (CLR_{ST}) property if there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Sz$,

for some $z \in S(X)$ and $z \in T(X)$.

Definition 2.11 [9] Two pairs (A,S) and (B,T) of self mappings of a fuzzy metric Space (X,M, *) are said to share CLRg of S property if there exist two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Sz$,

for some $z \in X$.

Proposition 2.1 [4] In a fuzzy metric space (X,M,*) limit of a sequence is unique.

Example 2.3 Let X = [0,∞) be the usual metric space . Define g , h : X→X by gx = x+ 3 and gx = 4x , for all $x \in X$. We consider the sequence $\{x_n\} = \{ 1 + 1/n \}$.

Since , $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = 4 = h(1) \in X$.

Therefore g and h satisfy the (CLRg) property.

Lemma 2.4 Let A, B , S and T be four self mapping of a fuzzy metric space (X,M,*) Satisfying following

1. The pair (A,S) (or (B,T)) satisfies the common limit in the range of S property (or T property)
2. There exists a constant $k \in (0,1)$ such that
   $(M(Ax,By,Kt))^2 \geq \min((M(Sx,Ty,t))^2, M(Sx,Ax,t), M(Sx,By,2t), M(Ty,Ax,t)$
   $M(Sx,By,2t), M(Ty,By,t))$, For all $x,y \in X$ and $t > 0$
3. $A(X) \subseteq T(X)$ (or $B(X) \subseteq S(X)$) .

Then the pairs (A,S) and (B,T) share the common limit in the range property.

Singh and Jain [8] proved the following results.

Theorem 2.1 Let A,B,S and T be self maps on a complete fuzzy metric space (X,M,*) Satisfying

1. $A(X) \subset T(X)$, $B(X) \subset T(X)$
2. One of A and B is continuous.
3. (A,S) is semi compatible and (B,T) is weak compatible.
4. For all $x,y \in X$ and $t > 0$
   $M(Ax,Bx,t) \geq r(M(Sx,Ty,t))$,

Where $r : [0,1]\to[0,1]$ is a continuous function such that $r(t) > t$, for each $0< t<1$. Then A,B,S and T have a unique common fixed point.

III. MAIN RESULT

In the following theorem we replace the continuity condition by using (CLRg) property.

Theorem 3.1 Let A, B, S and T be self mapping on a complete fuzzy metric space (X,M,*), where * is a continuous t – norm definied by ab = min [a,b] satisfying

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$.

(ii) (B,T) is semi compatible ,

(iii) Then for all $x,y \in X$ and $t > 0$.

$$M(Ax, By, kt) \geq \phi[\min(M(Sx, Ty, t), \{M(Sx,Ax,t) . M(By,Ty,t)\}, \tfrac{1}{2}(M(Ax, Ty,t) + M(By,Ax,t)]$$

Where $\phi : [0,1] \to [0,1]$ is a continuous function such that $\phi(1) = 1$, $\phi(0) = 0$ and $\phi(b) = b$, for $0 < b<1$.

If the pair (A,S) and (B, T) share the common limit in the range of S property , then A, B, S and T have a unique common fixed point

Proof –Let x_0 be any arbitrary point for which there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}, \text{ for } n= 0,1,2,\ldots$$

Now,

$$M(y_{2n+1},y_{2n+2},kt) = M(Ax_{2n},Bx_{2n+1},kt)$$
$$\geq \phi[\min(M((Sx_{2n},Tx_{2n+1}, t), \{M(Sx_{2n} A,x_{2n},t . M(Bx_{2n+1},Tx_{2n+!}, t)\}, \tfrac{1}{2}(M(Ax_{2n}, T_{2n+1},t) + M(Bx_{2n+1},Ax_{2n},t))]$$
$$\geq \phi[\min(M(y_{2n},y_{2n+1}, t), \{M(y_{2n}, y_{2n+1}, t) . M(y_{2n+2}, y_{2n+1}, t)\}, \tfrac{1}{2}(M(y_{2n+1}, y_{2n+1},t) + M(y_{2n+2},y_{2n+1},t))]$$
$$M(y_{2n+1},y_{2n+2},kt) > M(y_{2n}, y_{2n+1},t)$$

Similarly, we can proved $M(y_{2n+2}, y_{2n+3},t) > M(y_{2n+1},y_{2n+2},t)$

In general , $M(y_{n+1},y_n,t) > M(y_n,y_{n+1},t)$

Thus, from this we conclude that { $M(y_{n+1},y_n,t)$ is an increasing sequence of positive real numbers in [0,1] and tends to limit $l \leq 1$.

If $l < 1$, then $M(y_{n+1},y_n,t) \geq \phi(M(y_n, y_{n+1}, t)$,

Letting $n\to\infty$, we get $\lim_{n\to\infty} M(y_{n+1},y_n,t) \geq \phi[\lim_{n\to\infty} M(y_n, y_{n+1}, t)]$

$$l \geq \phi(l) = l \qquad (\text{Since } \phi(b) > b)$$

a contradiction . Now for any positive integer q

$$M(y_n,y_{n+q},t) \geq M(y_n, y_{n+1}, y_{n+q}, t/2(q-1)+1) * M(y_{n+1},y_{n+2}, y_{n+q}, t/2(q-1)+1)*\ldots* M(y_{n+q+1}, y_{n+q}, t/2(q-1)+1)$$

Taking limit, we get

$$\lim_{n\to\infty} M(y_n,y_{n+q},t) \geq \lim_{n\to\infty} M(y_n, y_{n+1}, y_{n+q}, t/2(q-1)+1) * l\lim_{n\to\infty} M(y_{n+1},y_{n+2}, y_{n+q}, t/2(q-1)+1)*\ldots* \lim_{n\to\infty} M(y_{n+q+!}, y_{n+q}, t/2(q-1)+1)$$

$$\operatorname{Lim}_{n\to\infty} M(y_n,y_{n+q},t) \geq 1*1*1*\ldots.*1=1$$

Which means $\{y_n\}$ is a Cauchy sequence in X . Since X is complete , then $y_n \to z$ in X.

That is $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$, $\{Bx_{2n+1}\}$ and $\{Sx_{2n}\}$ also converges to z in X.

Since, the pair (A,S) and (B,T) share the common limit in the range of S property , then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = Sz, , \text{ for some } z \in X.$$

First we prove that Az= Sz

By (3.3) , putting x= z and y= y_n , we get

$$M(Az,By_n,kt) \geq \emptyset[\min(M(Sz, Ty_n,t), \{M(Sz,Az,t) .M(By_n, Ty_n,t)\}, \tfrac{1}{2}(M(Az, Ty_n, t) + M(By_n, Az,t))]$$

Taking limit $n\to\infty$, we get

$$M(Az,Sz,kt) \geq \emptyset[\min(M(Sz, Sz, t), \{M(Sz,Az,t) . M(Sz,Sz,t)\}, \tfrac{1}{2}(M(Az,Sz,t) + M(Sz,Az,t))]$$
$$\geq \emptyset[\min(1, \{M(Sz, Az,t) . 1\}, M(Sz,Az,t)]$$
$$M(Az,Sz,kt) \geq M(Sz, Az,t)$$

Hence by Lemma 2.2 , we get Az= Sz …(1)

Since , $A(X) \subseteq T(X)$, therefore there exist $u \in X$, such that Az=Tu …(2)

Again , by inequality (iii), putting x=z and y= u , we get

$$M(Az,Su,kt) \geq \emptyset[\min(M(Sz, Tu, t), \{M(Sz,Az,t) . M(Bu,Tu,t)\}, \tfrac{1}{2}(M(Az,Tu,t) + M(Bu,Az,t))]$$

Using (1) and (2) , we get

$$M(Tu, Bu,kt) \geq \emptyset[\min(M(Az, Tu, t), \{M(Az,Az,t) . M(Bu, Tu, t)\}, \tfrac{1}{2}(M(Az,Az,t) + M(Bu, Tu,t))]$$

$\geq \emptyset$ [min M(Tu,Tu,t) , { 1. M(Bu, Tu,t) } , M (Bu,Tu,t)]

M(Tu,Bu,kt) $\geq$ M (Bu, Tu,t)

Hence , by Lemma 2.2 , we get Tu=Bu …(3)

Thus , from (1),(2) and (3) , we get Az= Sz= Tu = Bu …(4)

Now , we will prove that Az= z

By inequality(iii), putting x= z and y = x_{2n+1} , we get

M(Az,B$x_{2n+!}$,kt) $\geq \emptyset$ [min (M(Sz,Tx_{2n+1},t) , { M(Sz,Az,t) . M(Bx_{2n+1},Tx_{2n+1},t) }

½(M(Az, Tx_{2n+1},t) + M(Bx_{2n+1},Az,t))]

Taking limit n $\to\infty$, using (1) , we get

M(Az,z,t) $\geq \emptyset$ [min (M(Sz,z,t), { M(Az,Az,t) . M(z,z,t) }, ½ (M(Az,z,t)+ M(z,Az,t))]

M(Az,z,t) $\geq \emptyset$ [min (M(Az,z,t), { 1,1} , M(Az,z,t)]

M(Az,z,t) $\geq$ M(Az,z,t)

Hence , by Lemma 2.2 , we get Az=z

Thus , from (4), we get z= Tu=Bu

Now , Semi compatibility of (B,T) gives BTy_{2n+1} $\to$Tz , i.e. Bz=Tz.

Now, putting x=z and y= z in inequality (iii), we get

M(Az, Bz,t) $\geq \emptyset$ [min (M(Sz,Tz,t) , { M (Sz,Az,t) .M(Bz,Tz,t) } ,

1/2 (M(Az,Tz,t)+ M(Bz,Az,t))]

M(Az, Bz,t) $\geq \emptyset$ [min (M(Az,Bz,t) , { M (Az,Az,t) .M(Tz,Tz,t) } ,

1/2 (M(Az,Bz,t)+ M(Bz,Az,t))]

M(Az,Bz,t) $\geq$ M(Az,Bz,t)

Hence , by Lemma 2.2 , we get Az=Bz.

And, hence Az= Bz = z.

Combining ,all result we get z= Az=Bz=Sz=Tz.

From, this we conclude that z is a common fixed point of A,B,S and T.

Uniquness

Let z_1 be another common fixed point of A,B,S and T. Then

z_1 = Az_1 = Bz_1 =Sz_1 = Tz_1 , and z =Az=Bz=Sz=Tz

Then, by inequality (iii) , putting x=z and y =z_1,we get

M(z,z_1,kt) = M(Az,Bz_1,t) $\geq \emptyset$ [min (M(Sz, Tz_1, t), {M(Sz,Az,t) . M(Bz_1,Tz_1,t)},

½ (M(Az, Tz_1,t) + M(Bz_1,Az,t))]

$\geq \emptyset$ [min (M (z,z_1,t) , { M(z,z,t) . M(z_1,z_1,t) },

½ (M(z,z_1,t) + M(z_1,z,t))]

$\geq \emptyset$ [min (M(z,z_1,t) , 1, M(z,z_1,t)

M(z,z_1,t) $\geq$ M(z,z_1,t)

Hence, from Lemma 2.2 , we get z=z_1

Thus z is the unique common fixed point of A, B, S and T.

Corollary 3.2 Let (X,M,*) be complete fuzzy metric space . suppose that the mapping A, B,S and T are self maps of X satisfying(i-ii) conditions and there exist k$\in$ (0,1) such that

M(Ax,By,kt) $\geq$ M(Sx,Ty,t), M(Ax,Sx,t), M(By,Ty,t), M(By,Sx,2t), M(Ax,Ty,t)

For every x,y $\in$ X , t> 0 . Then A,B,S and T have a unique common fixed point in X.

Corollary3.3 Let (X,M,*) be complete fuzzy metric space . suppose that the mapping A, B,S and T are self maps of X satisfying (i-ii) conditions and there exist k$\in$ (0,1) such that

M(Ax,By,kt) $\geq$ M(Sx,Ty,t), M(Sx,Ax,t), M(Ax,Ty,t)

For every x,y $\in$ X , t> 0 . Then A,B,S and T have a unique common fixed point in X.

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