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Fixed Point Theorem Satisfying (ξ,η) Contractive Condition in Complete G-Metric Space

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Abstract: *The purpose of the present paper is to prove a unique fixed point theorem for a self mapping satisfying (ξ,η) contractive condition in partially ordered complete G-metric space. As an application, the existence and uniqueness of the solution of initial value problem for the non homogeneous heat equation in one dimension has been discussed.*

Keywords: *Altering distance function , complete G-metric space , Fixed point , G -Cauchy sequence, initial value problems , (ξ,η) contractive condition.*

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I. Introduction

Fixed point theory has been one of the most rapidly developing fields in analysis during the last few decades. It is well known that the contractive-type conditions are very indispensable in the study of fixed point theory. The first important result on fixed points for contractive-type mappings was the well-known Banach - Caccioppoli theorem which was published in 1922. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics.

One of the most common applications of the fixed point theory is the problem of existence and uniqueness of solutions of initial and boundary value problems for differential and integral equations. The number of studies dealing with such problems has increased considerably in the recent years.

In 2006, Mustafa in collaboration with Sims introduced a new notion of generalized metric space called G -metric space [10]. In fact, Mustafa et al. studied many fixed point results for a self-mapping in G -metric space under certain conditions, see [9,10, 11, 12, 13]. For other results on G -metric spaces, see [14,15,16,17]. In the present work, we study some fixed point result for a self-mapping in a partially ordered complete G - metric space X satisfying (ξ,η) contractive condition with its application to solve the initial value problem.

Following preliminaries and basic definitions are used through-out the paper.

Definition 1.1: Let X be a non empty set, and let $G: X \times X \times X \to R^+$ be a function satisfying the following properties:

(G_1) $G(x,y,z)=0$ if $x=y=z$

(G_2) $0<G(x,x,y)$ for all $x,y\in X$,with $x\neq y$

(G_3) $G(x,x,y)\leq G(x,y,z)$, for all $x,y,z\in X$, with $y\neq z$

(G_4) $G(x,y,z)=G(x,z,y)=G(y,z,x)$ (Symmetry in all three variables)

(G_5) $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$, for all $x,y,z,a\in X$ (rectangle inequality)

Then the function G is called a generalized metric , or more specially a G - metric on X , and the pair (X,G) is called a G −metric space.

Definition 1.2: Let (X,G) be a G - metric space and let $\{x_n\}$ be a sequence of points of X , a point $x\in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to+\infty} G(x,x_n,x_m)=0$, and we say that the sequence $\{x_n\}$ is G - convergent to x or $\{x_n\}$ G -converges to x .

Thus, $x_n \to x$ in a G - metric space (X,G) if for any $\in > 0$ there exists $k \in N$ such that $G(x, x_n, x_m) < \in$, for all $m,n \ge k$

Proposition 1.3: Let (X,G) be a G - metric space. Then the following are equivalent:

i) $\{x_n\}$ is G - convergent to x

ii) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$

iii) $G(x_n, x, x) \to 0$ as $n \to +\infty$

iv) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$

Proposition 1.4 : Let (X,G) be a G - metric space. Then for any x ,y , z , a in X it follows that

i) If $G(x,y,z) = 0$ then $x = y = z$

ii) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$

iii) $G(x,y,y) \le 2G(y,x,x)$

iv) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$

v) $G(x,y,z) \le \frac{2}{3}\left(G(x,y,a) + G(x,a,z) + G(a,y,z)\right)$

vi) $G(x,y,z) \le \left(G(x,a,a) + G(y,a,a) + G(z,a,a)\right)$

Definition 1.5: Let (X,G) be a G - metric space. A sequence $\{x_n\}$ is called a G - Cauchy sequence if for any $\in > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_l) < \in$ for all $m,n,l \ge k$, that is $G(x_n, x_m, x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 1.6: Let (X,G) be a G - metric space .Then the following are equivalent:

i) The sequence $\{x_n\}$ is G - Cauchy;

ii) For any $\in > 0$ there exists $k \in N$ such that $G(x_n, x_m, x_m) < \in$ for all $m,n \ge k$

Proposition 1.7: A G - metric space (X,G) is called G -complete if every G -Cauchy sequence is G - convergent in (X,G).

Definition1.8: A function $\phi : [0,\infty) \to [0,\infty)$ is said to an altering distance function if it is continuous , non-decreasing and $\phi(t) = 0$ if and only if $t = 0$

II. Main Result

Theorem 2.1: Let $(X, G, \le)$ be a complete G-metric space and $T : X \to X$ be a self map satisfying

$$\xi\left(G\left(Tx,Ty,T^2x\right)\right) \le \xi\left(\mu(x,y,Tx)\right) - \eta\left(\mu(x,y,Tx)\right) + P\left(m(x,y,Tx)\right) \tag{2.1.1}$$

for all $x, y \in X$ with $x \le y$; $\xi, \eta : [0,\infty) \to [0,\infty)$ are both continuous , non-decreasing functions with $\xi(t) = 0 = \eta(t)$ if and only if $t = 0$ and $\xi(t) < t$ for all $t > 0$. Let $P > 0$

also $\mu(x,y,Tx) = \max.\left\{G(x,y,Tx), G(x,Tx,Ty), G\left(y,Ty,T^2x\right), G\left(Tx,T^2x,Tx\right)\right\}$

$$m(x,y,Tx) = \min.\begin{cases} G(x,y,Tx), G(x,Tx,Ty), G\left(y,Ty,T^2x\right), G\left(Tx,T^2x,Tx\right), \\ G(x,Ty,Ty), G(y,Tx,Tx), G(Tx,Ty,Ty) \end{cases}$$

If there exists $x_0 \in X$ such that $x_0 \le Tx_0$ then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\}$ as $x_n = Tx_{n-1}$.

Let $x_n \ne x_{n+1} \ne x_{n+2} = Tx_{n+1}$

By setting $x = x_{n-1}$, $y = x_n$, $Tx = x_{n+1}$ in (2.1.1) ,

$$\xi\left(G\left(Tx_{n-1}, Tx_n, Tx_{n+1}\right)\right) = \xi\left(G\left(x_n, x_{n+1}, x_{n+2}\right)\right)$$
$$\le \xi\left(\mu\left(x_{n-1}, x_n, x_{n+1}\right)\right) - \eta\left(\mu\left(x_{n-1}, x_n, x_{n+1}\right)\right) + P\left(m\left(x_{n-1}, x_n, x_{n+1}\right)\right) \tag{2.1.2}$$

where

$$m(x_{n-1},x_n,x_{n+1}) = \min.\begin{Bmatrix} G(x_{n-1},x_n,x_{n+1}), G(x_{n-1},Tx_{n-1},Tx_n), G(x_n,Tx_n,Tx_{n+1}), G(x_{n+1},Tx_{n+1},Tx_{n-1}), \\ G(x_{n-1},Tx_n,Tx_n), G(x_n,Tx_{n-1},Tx_{n-1}), G(x_{n+1},Tx_n,Tx_n) \end{Bmatrix}$$

$$= 0$$

$$\mu(x_{n-1},x_n,x_{n+1}) = \max.\{G(x_{n-1},x_n,x_{n+1}), G(x_{n-1},Tx_{n-1},Tx_n), G(x_n,Tx_n,Tx_{n+1}), G(x_{n+1},Tx_{n+1},Tx_{n-1})\}$$

$$= \max.\{G(x_{n-1},x_n,x_{n+1}), G(x_{n-1},x_n,x_{n+1}), G(x_n,x_{n+1},x_{n+2}), G(x_{n+1},x_{n+2},x_n)\}$$
$$= \max.\{G(x_{n-1},x_n,x_{n+1}), G(x_n,x_{n+1},x_{n+2})\}$$

If $\mu(x_{n-1},x_n,x_{n+1}) = G(x_n,x_{n+1},x_{n+2})$, then

$\xi(G(x_n,x_{n+1},x_{n+2})) \le \xi(G(x_n,x_{n+1},x_{n+2})) - \eta(G(x_n,x_{n+1},x_{n+2}))$, which implies that

$\eta(G(x_n,x_{n+1},x_{n+2})) = 0$ and hence, $G(x_n,x_{n+1},x_{n+2}) = 0$

i.e. $x_n = x_{n+1} = x_{n+2}$, which is a contradiction to the initial assumption.

Therefore only possibility is that $\mu(x_{n-1},x_n,x_{n+1}) = G(x_{n-1},x_n,x_{n+1})$

Therefore $\xi(G(x_n,x_{n+1},x_{n+2})) \le \xi(G(x_{n-1},x_n,x_{n+1})) - \eta(G(x_{n-1},x_n,x_{n+1}))$ (2.1.3)

i.e. $\xi(G(x_n,x_{n+1},x_{n+2})) \le \xi(G(x_{n-1},x_n,x_{n+1}))$

Since ξ is non-decreasing , therefore $G(x_n,x_{n+1},x_{n+2}) \le G(x_{n-1},x_n,x_{n+1})$, for $n \ge 1$.

i.e. The sequence $\{G(x_n,x_{n+1},x_{n+2})\}$ is decreasing and positive. Therefore it will converge to some positive number say $r > 0$.

Therefore taking limit as $n \to \infty$ in (2.1.3) implies that $\xi(r) \le \xi(r) - \eta(r)$,

it implies $\eta(r) = 0$ and hence $r = 0$.

$\therefore \lim_{n\to\infty} G(x_n,x_{n+1},x_{n+2}) = 0$ (2.1.4)

Also by using (G_3) one can write , $\lim_{n\to\infty} G(x_n,x_{n+1},x_{n+1}) = 0$ (2.1.5)

Now, to show that $\{x_n\}$ is a G-Cauchy sequence.

On the contrary assume that , $\{x_n\}$ is not a G-Cauchy sequence.

Therefore there exists an $\in > 0$ for which the subsequences $\{x_{m(i)}\}$ and $\{x_{n(i)}\}$ of $\{x_n\}$ can be obtained with $n(i) > m(i) > i$ such that $G(x_{n(i)},x_{m(i)},x_{m(i)}) \ge \in$ (2.1.6)

Also corresponding to $m(i)$, one can find $n(i)$ in such a way that it is the smallest integer with $n(i) > m(i)$ and satisfying (2.1.4) then $G(x_{n(i-1)},x_{m(i)},x_{m(i)}) < \in$ (2.1.7)

By using (2.1.6) and rectangular inequality ,it can be written that

$$\in \le G(x_{n(i)},x_{m(i)},x_{m(i)}) \le G(x_{n(i)},x_{n(i-1)},x_{n(i-1)}) + G(x_{n(i-1)},x_{m(i)},x_{m(i)})$$
$$< G(x_{n(i)},x_{n(i-1)},x_{n(i-1)}) + \in \quad (2.1.8)$$

Also , $0 \le G(x_{n(i)},x_{n(i-1)},x_{n(i-1)}) = G(x_{n(i-1)},x_{n(i-1)},x_{n(i)})$

Applying limit as $i \to \infty$ and using (2.1.5) , $G(x_{n(i-1)},x_{n(i-1)},x_{n(i)}) \to 0$ (2.1.9)

Therefore by using (2.1.8) we have , $\lim_{i\to\infty} G(x_{n(i)},x_{m(i)},x_{m(i)}) = \in$ (2.1.10)

Again by using rectangular inequality, one can write

$$G(x_{n(i)},x_{m(i)},x_{m(i)}) \le G(x_{n(i)},x_{n(i-1)},x_{n(i-1)}) + G(x_{n(i-1)},x_{m(i-1)},x_{m(i-1)}) + G(x_{m(i-1)},x_{m(i)},x_{m(i)})$$
$$G(x_{n(i-1)},x_{m(i-1)},x_{m(i-1)}) \le G(x_{n(i-1)},x_{n(i)},x_{n(i)}) + G(x_{n(i)},x_{m(i)},x_{m(i)}) + G(x_{m(i)},x_{m(i-1)},x_{m(i-1)})$$

Applying limit as $i \to \infty$ and using (2.1.9) , (2.1.10),

$$\lim_{i\to\infty} G\left(x_{n(i-1)}, x_{m(i-1)}, x_{m(i-1)}\right) = \in \quad (2.1.11)$$

Now , by using (2.1.2) , (2.1.6) and (2.1.7) ,

$$\begin{aligned}\xi\left(G\left(Tx_{n(i-1)}, Tx_{m(i-1)}, T^2x_{n(i-2)}\right)\right) &= \xi\left(G\left(x_{n(i)}, x_{m(i)}, x_{n(i)}\right)\right)\\ &\le \xi\left(\mu\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-2)}\right)\right) - \eta\left(\mu\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-2)}\right)\right)\\ &\quad + P\left(m\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-2)}\right)\right)\end{aligned} \quad (2.1.12)$$

where

$$m\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-2)}\right) = \min.\left\{\begin{array}{l} G\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-1)}\right), G\left(x_{n(i-1)}, Tx_{n(i-1)}, Tx_{m(i-1)}\right),\\ G\left(x_{m(i-1)}, Tx_{m(i-1)}, T^2x_{n(i-2)}\right), G\left(Tx_{n(i-1)}, T^2x_{n(i-2)}, Tx_{n(i-1)}\right),\\ G\left(x_{n(i-1)}, Tx_{m(i-1)}, Tx_{m(i-1)}\right), G\left(x_{m(i-1)}, Tx_{n(i-1)}, Tx_{n(i-1)}\right),\\ G\left(Tx_{n(i-1)}, Tx_{m(i-1)}, Tx_{m(i-1)}\right)\end{array}\right\}$$

For limit as $i \to \infty$ and using (2.1.9) , $\lim_{i\to\infty} m\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-2)}\right) = 0$ (2.1.13)

$$\mu\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-2)}\right) = \max.\left\{\begin{array}{l} G\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-1)}\right), G\left(x_{n(i-1)}, Tx_{n(i-1)}, Tx_{m(i-1)}\right),\\ G\left(x_{m(i-1)}, Tx_{m(i-1)}, T^2x_{n(i-2)}\right), G\left(Tx_{n(i-1)}, T^2x_{n(i-2)}, Tx_{n(i-1)}\right)\end{array}\right\}$$

$$= \max.\left\{\begin{array}{l} G\left(x_{n(i-1)}, x_{m(i-1)}, x_{n(i)}\right), G\left(x_{n(i-1)}, x_{n(i)}, x_{m(i)}\right),\\ G\left(x_{m(i-1)}, x_{m(i)}, x_{n(i)}\right), G\left(x_{n(i)}, x_{n(i)}, x_{n(i)}\right)\end{array}\right\}$$

Applying limit as $i \to \infty$, using (2.1.9) and (2.1.10) ,

$$\lim_{i\to\infty} \mu\left(x_{n(i-1)}, x_{m(i-1)}, Tx_{n(i-2)}\right) = \{\in, \in, \in, 0\} = \in \quad (2.1.14)$$

Therefore from (2.1.12) as $i \to \infty$, using (2.1.13) and (2.1.14) , we get $\xi(\in) \le \xi(\in) - \eta(\in)$, it implies that $\eta(\in) = 0$ i.e. $\in = 0$, which is a contradiction as $\in > 0$.

Therefore $\{x_n\}$ is a G-Cauchy sequence.

Since X is a complete G-metric space , $\therefore \{x_n\}$ converges to some $u \in X$.

Therefore $\lim_{n\to\infty} G(x_n, x_n, u) = \lim_{n\to\infty} G(x_n, u, u) = 0$ (2.1.15)

Now , to show that u is fixed point of T.

Replacing $x = u$, $y = x_{n+1}$, $Tx = x_{n+1}$ in (2.1.1) ,

$$\xi\left(G\left(Tu, Tx_{n+1}, Tx_{n+1}\right)\right) \le \xi\left(\mu\left(u, x_{n+1}, Tu\right)\right) - \eta\left(\mu\left(u, x_{n+1}, Tu\right)\right) + P\left(m\left(u, x_{n+1}, Tu\right)\right) \quad (2.1.16)$$

where

$$m\left(u, x_{n+1}, Tu\right) = \min.\left\{\begin{array}{l} G\left(u, x_{n+1}, Tu\right), G\left(u, Tu, Tx_{n+1}\right), G\left(x_{n+1}, Tx_{n+1}, Tx_{n+1}\right), G\left(Tu, Tx_{n+1}, Tu\right),\\ G\left(u, Tx_{n+1}, Tx_{n+1}\right), G\left(x_{n+1}, Tu, Tu\right), G\left(Tu, Tx_{n+1}, Tx_{n+1}\right)\end{array}\right\}$$

Taking limit as $n \to \infty$ implies $\lim_{n\to\infty} m\left(u, x_{n+1}, Tu\right) = 0$

$$\mu(x, y, Tx) = \max.\left\{G(x, y, Tx), G(x, Tx, Ty), G\left(y, Ty, T^2x\right), G\left(Tx, T^2x, Tx\right)\right\}$$

Also , $\mu\left(u, x_{n+1}, Tu\right) = \max.\left\{G\left(u, x_{n+1}, Tu\right), G\left(u, Tu, Tx_{n+1}\right), G\left(x_{n+1}, Tx_{n+1}, Tx_{n+1}\right), G\left(Tu, Tx_{n+1}, Tu\right)\right\}$

Taking limit as $n \to \infty$, $\lim_{n\to\infty} \mu\left(u, x_{n+1}, Tu\right) = \max\{G(u, u, Tu), G(u, Tu, Tu)\}$

$\le G(Tu, u, u) + G(u, Tu, u)$

As $n \to \infty$ from (2.1.16) , $\xi(G(u, u, Tu)) \le \xi(G(u, u, Tu)) - \eta(G(u, u, Tu)) + 0$

i.e. $\xi(G(u,u,Tu)) \le \xi(G(u,u,Tu)) - \eta(G(u,u,Tu))$, it implies that $\eta(G(u,u,Tu)) = 0$
i.e. $G(u,u,Tu) = 0$ i.e. $Tu = u$
Hence u is fixed point of T.
Now , to prove the uniqueness of u.
If possible , assume that v is another fixed point of T.
Therefore consider , $\xi(G(u,u,Tv)) = \xi(G(Tu,Tu,Tv))$

$$\le \xi(\mu(u,u,v)) - \eta(\mu(u,u,v)) + P(m(u,u,v)) \qquad (2.1.17)$$

where

$$m(u,u,v) = \min.\{G(u,u,Tu), G(u,Tu,Tu), G(u,Tu,Tv), G(Tu,Tv,Tu), G(u,Tu,Tu), G(u,Tu,Tu), G(v,Tu,Tu)\}$$

Therefore $m(u,u,v) = 0$

$$\mu(x,y,Tx) = \max.\{G(x,y,Tx), G(x,Tx,Ty), G(y,Ty,T^2x), G(Tx,T^2x,Tx)\}$$

and $\mu(u,u,v) = \max.\{G(u,u,Tu), G(u.Tu,Tu), G(u,Tu,Tv), G(Tu,Tv,Tu)\}$

$$= \max.\{G(u,u,v), G(v,v,u)\}$$

$$= G(u,u,v) \; (\because \; G(v,v,u) \le G(v,u,u,) + G(u,v,u))$$

$\therefore \; \xi(G(u,u,v)) \le \xi(G(u,u,v)) - \eta(G(u,u,v)) + 0$, it gives $\eta(G(u,u,v)) = 0$
i.e. $G(u,u,v) = 0$, and hence it implies that $u = v$.
Hence u is the unique fixed point of T.

III. Application

As an application of the Theorem 2.1, consider the problem of existence and uniqueness of an initial value problem defined by a non linear heat equation in one dimension. Such an initial value problem is defined as follows:

$$y_t(x,t) = y_{xx}(x,t) + Y(x,t,u,y_x) \; , \quad -\infty < x < \infty \; , \; 0 < t < T$$

$$y(x,0) = \phi(x) \qquad (3.1)$$

where ϕ is assumed to be continuously differentiable, ϕ and ϕ' bounded , $Y(x,t,y,y_x)$ is continuous function.

Definition 3.2: A solution of the initial value problem (3.1) is any function $y = y(x,t)$ defined in $R \times I$, where $I = (0,T]$, C is the set of all continuous functions defined in $R \times I$ satisfying the equation and the condition in (3.1) and also the conditions:

(i) $y \in C(R \times I)$

(ii) y_t , y_x and $y_{xx} \in C(R \times I)$

(iii) y and y_x are bounded in $R \times I$

Consider the space X defined as ,$X = \{u(x,t) : u, u_x \in C(R \times I) \text{ and } \|u\| < \infty\}$

where the norm on this space is defined as , $\|u\| = \sup_{x\in R, t\in I} |u(x,t)| + \sup_{x\in R, t\in I} |u_x(x,t)|$ (3.3)

The set X endowed with the norm $\|.\|$ defined in (3.3) is a Banach space. Define a *G*-metric on X as follows:

$$G(u,v,w) = \sup_{x\in R,t\in I} |u(x,t) - v(x,t)| + \sup_{x\in R,t\in I} |u_x(x,t) - v_x(x,t)| + \sup_{x\in R,t\in I} |v(x,t) - w(x,t)|$$

$$+ \sup_{x\in R,t\in I} |v_x(x,t) - w_x(x,t)| + \sup_{x\in R,t\in I} |u(x,t) - w(x,t)| + \sup_{x\in R,t\in I} |u_x(x,t) - w_x(x,t)|$$

Then (X,G) is a complete G-metric space . Define also a partial order $\le$ on X as
$u, v \in X$, $u \le v \Leftrightarrow u(x,t) \le v(x,t)$, $u_x(x,t) \le v_x(x,t)$, for any $x \in R$ and $t \in I$.

It can be easily verified that every pair of elements in X has a lower bound or an upper bound. For any $u,v \in$ X max$\{u,v\}$ and min$\{u,v\}$ are the lower and upper bounds for *u* and *v* respectively. Let $\{v_n\} \subseteq$ X be a monotone non-decreasing sequence which converges to *v* in X .
Then, for any $x \in R$ and $t \in I$, we have
$v_1(x,t) \le v_2(x,t) \le --- \le v_n(x,t) \le ---$ and $v_{1x}(x,t) \le v_{2x}(x,t) \le --- \le v_{nx}(x,t) \le ---$
Moreover, since the sequences $\{v_n(x,t)\}$ and $\{v_{nx}(x,t)\}$ of real numbers converge to $v(x,t)$ and $v_x(x,t)$ respectively , $\therefore$ for all $x \in R,\ t \in I$ and $n \ge 1$ the inequalities $v_n(x,t) \le v(x,t)$ and $v_{nx}(x,t) \le v_x(x,t)$ hold. Therefore $v_n \le v$ for all $n \ge 1$ and hence the set (X,$\le$) with the G-metric defined above satisfies $v_n \le v$, for all $n \ge 1$.

Definition 3.4: A lower solution of the initial value problem (3.1) is a function $y \in$ X such that

$$y_t(x,t) \le y_{xx}(x,t) + Y(x,t,y,y_x)\ , \quad -\infty < x < \infty\ ,\ 0 < t < T$$
$$y(x,0) \le \phi(x) \quad , \quad -\infty < x < \infty$$

where the function ϕ is continuously differentiable , both ϕ and ϕ' are bounded , the set X is the set defined above and $Y(x,t,y,y_x)$ is continuous function.
Consider the following theorem for the solution of the initial value problem (3.1).

Theorem 3.5: Consider the problem (3.1) and, assume that $Y : R \times I \times R \times R \to R$ is a continuous function. Suppose that the following conditions hold:

(1) For any $\alpha > 0$, the function $Y(x,t,s,p)$, where $|s| < \alpha$ and $|p| < \alpha$ is uniformly continuous in *x* and *t* , for each compact subset of $R \times I$.

(2) There exists a constant $\alpha_1 \le \frac{1}{3}\left(T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)^{-1}$, such that

$$0 \le Y(x,t,s_2,p_2) - Y(x,t,s_1,p_1) \le \alpha_1\, Inf(s_2 - s_1 + p_2 - p_1 + 1)$$

for all (s_1,p_1) and (s_2,p_2) in $R \times R$ with $s_1 \le s_2$ and $p_1 \le p_2$.

(3) Y is bounded for bounded *s* and *p*.

Then the existence of a lower solution for the initial value problem (3.1) provides the existence of the unique solution of the problem (3.1).

Proof: It is clear that the problem (3.1) is equivalent to the integral equation

$$y(x,t) = \int_{-\infty}^{\infty} k(x-\zeta,t)\phi(\zeta)\,d\zeta + \int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)Y(\zeta,\tau,y(\zeta,t),y_\zeta(\zeta,\tau))\,d\zeta\,d\tau$$

for all $x \in R$ and $0 < t \le T$, where the function $k(x,t)$ is the Green's function of the problem defined as

$$k(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left\{\frac{-x^2}{4t}\right\}$$, for all $x \in R$ and $0 < t$.

The initial value problem (3.1) has a unique solution if and only ifthe above integral equation has unique solution *y* such that *y* and y_x are continuous and bounded for all $x \in$ R and 0<$t \le T$.
Define a mapping $f : X \to X$ by

$$(f\ y)(x,t) = \int_{-\infty}^{\infty} k(x-\zeta,t)\phi(\zeta)\,d\zeta + \int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)Y(\zeta,\tau,u(\zeta,\tau),y_\zeta(\zeta,\tau))\,d\zeta\,d\tau$$

for all $x \in R$ and $t \in I$.
The fixed point $y \in$ X of a function f is a solution of the problem (3.1) , where f is non-decreasing as by condition (2) of Theorem 3.5 for $y \ge z$ and $y_x \ge z_x$, one can write
$Y(x,t,y(x,t),y_x(x,t)) \ge Y(x,t,z(x,t),z_x(x,t))$

Since $k(x,t) > 0$ for all $(x,t) \in R \times I$,

$$(f\,y)(x,t) = \int_{-\infty}^{\infty} k(x-\zeta,t)\phi(\zeta)\,d\zeta + \int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)Y(\zeta,\tau,y(\zeta,t),y_\zeta(\zeta,t))\,d\zeta\,d\tau$$

$$\geq \int_{-\infty}^{\infty} k(x-\zeta,t)\phi(\zeta)\,d\zeta + \int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)Y(\zeta,\tau,z(\zeta,t),z_\zeta(\zeta,t))\,d\zeta\,d\tau$$

$$= (f\,z)(x,t)$$

i.e. $(f\,y)(x,t) \geq (f\,z)(x,t)$, for all $x \in R$ and $t \in I$.

Also, $\left|(f\,y)(x,t) - (f\,z)(x,t)\right|$

$$\leq \int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)\left|Y(\zeta,\tau,y(\zeta,\tau),y_\zeta(\zeta,\tau)) - Y(\zeta,\tau,z(\zeta,\tau),z_\zeta(\zeta,\tau))\right|d\zeta\,d\tau$$

$$\leq \int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)\,\alpha_1 \inf(y(\zeta,\tau) - z(\zeta,\tau) + y_\zeta(\zeta,\tau)) - z_\zeta(\zeta,\tau) + 1)\,d\zeta\,d\tau$$

$$\leq \alpha_1 \inf(G(y,z,fy)+1)\int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)\,d\zeta\,d\tau$$

$$\leq \alpha_1 \inf(G(y,z,fy)+1)T \tag{3.6}$$

where $\inf(y(\zeta,\tau) - z(\zeta,\tau) + y_\zeta(\zeta,\tau) - z_\zeta(\zeta,\tau) + 1)$

$$\leq \inf(2 \sup_{\zeta\in R,\tau\in I}\left|y(\zeta,\tau) - z(\zeta,\tau)\right| + 2 \sup_{\zeta\in R,\tau\in I}\left|y_\zeta(\zeta,\tau) - z_\zeta(\zeta,\tau)\right| + 1)$$

$$= \inf(G(y,z,fy)+1) \tag{3.7}$$

and $\int_0^t \int_{-\infty}^{\infty} k(x-\zeta,t-\tau)\,d\zeta\,d\tau = T$ (3.8)

Since every pair of elements of X having lower bound or an upper bound and for every $x, y \in$ X there exists $z \in$ X which is comparable to both x and y. Therefore , either z or fy are comparable or there exists some $u \in$ X which is comparable to both z and fy.

In either case , it can be shown that

$$\left|(fz)(x,t) - (f^2 y)(x,t)\right| \leq \alpha_1 \inf(G(y,z,fy)+1)T \tag{3.9}$$

and $\left|(fy)(x,t) - (f^2 y)(x,t)\right| \leq \alpha_1 \inf(G(y,z,fy)+1)T$ for all $y \geq z$ (3.10)

By using differentiation under integral sign ,

$$\left|\frac{\partial fy}{\partial x}(x,t) - \frac{\partial fz}{\partial x}(x,t)\right| \leq \alpha_1 \inf(G(y,z,fy)+1)\int_0^t \int_{-\infty}^{\infty}\left|\frac{\partial k}{\partial x}(x-\zeta,t-\tau)\right|d\zeta\,d\tau$$

$$\leq \alpha_1 \inf(G(y,z,fy)+1)2\pi^{-\frac{1}{2}}T^{-\frac{1}{2}} \tag{3.11}$$

$$\left|\frac{\partial fz}{\partial x}(x,t) - \frac{\partial f^2 y}{\partial x}(x,t)\right| \leq \alpha_1 \inf(G(y,z,fy)+1)\int_0^t \int_{-\infty}^{\infty}\left|\frac{\partial k}{\partial x}(x-\zeta,t-\tau)\right|d\zeta\,d\tau$$

$$\leq \alpha_1 \inf(G(y,z,fy)+1)2\pi^{-\frac{1}{2}}T^{-\frac{1}{2}} \tag{3.12}$$

$$\left|\frac{\partial fy}{\partial x}(x,t) - \frac{\partial f^2 y}{\partial x}(x,t)\right| \leq \alpha_1 \inf(G(y,z,fy)+1)\int_0^t \int_{-\infty}^{\infty}\left|\frac{\partial k}{\partial x}(x-\zeta,t-\tau)\right|d\zeta\,d\tau$$

$$\leq \alpha_1 \inf(G(y,z,fy)+1)2\pi^{-\frac{1}{2}}T^{-\frac{1}{2}} \quad (3.13)$$

Using (3.6) , (3.9) and (3.10) with (3.11) , (3.12) and (3.13) , it is concluded that

$$G(fy,fz,f^2y) \leq 3\alpha_1(T+2\pi^{-\frac{1}{2}}T^{-\frac{1}{2}})\inf(G(y,z,fy)+1)$$
$$\leq \inf(G(y,z,fy)+1) \quad (3.14)$$

Define $\xi,\eta:[0,\infty)\to[0,\infty)$ both continuous , non-decreasing functions with $\xi(t)=0=\eta(t)$ if and only if $t=0$ and $\xi(t)<t$ for all $t>0$.

From (3.14) , $\xi(G(fy,fz,f^2y)) < G(fy,fz,f^2y)$

i.e. $\xi(G(fy,fz,f^2y)) \leq \xi(\mu(y,z,fy)) - \eta(\mu(y,z,fy)) + P(m(y,z,fy))$ is the contractive condition of Theorem 2.1 and μ , m are having same value as Theorem 2.1.

If $a \leq fa$ then $a(x,t)$ is a lower solution of (3.1).

For $-\infty<\zeta<\infty$ and $0<\tau<t$, we have

$$a(x,t) \leq \int_{-\infty}^{\infty} k(x-\zeta,t)\phi(\zeta)\,d\zeta + \int_0^t\int_{-\infty}^{\infty} k(x-\zeta,t-\tau)Y(\zeta,\tau,a(\zeta,\tau),a_\zeta(\zeta,\tau))\,d\zeta\,d\tau$$

$= (fa)(x,t)$, for all $x\in R$ and $t\in(0,T]$

Therefore by Theorem 2.1 f has a unique fixed point. It completes the proof.

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