International Journal of Engineering Science Invention (IJESI)
ISSN (Online): 2319 – 6734, ISSN (Print): 2319 – 6726
www.ijesi.org ||Volume 7 Issue 1|| January 2018 || PP. 33-43

Wave Free Potentials in the Theory of Water Waves Having Free Surface Boundary Condition with Higher Order Derivatives

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***Abstract**: The method of constructing wave-free potentials in a systematic manner for a number of situations such as two-dimensional non-oblique and oblique waves, three dimensional waves in a fluid with free surface condition with higher order partial derivative are presented here. In particular, these are obtained taking into account of the effect of the presence of free surface, surface tension at the free surface and in the presence of an ice-cover modelled as thin elastic plate.*

***Key Words:** wave-free potentials, non-oblique and oblique wave, free surface, ice-cover, Laplace operator.*

Date of Submission: 21-01-2018 Date of acceptance: 05-02-2018

I. Introduction

Problems involving generation or scattering of surface water waves by a body of any geometrical configuration present in water are of immense importance in ocean related industry and are generally investigated mathematically assuming linear theory. The problem of heaving motion of a long, horizontal circular cylinder on the surface of water was investigated by Ursell (1949) using the method of multipole expansion of the time-harmonic stream function. The corresponding velocity potential also has a similar expansion. Infact, for an infinitely long horizontal cylinder of arbitrary cross section floating on the surface of water, the potential function in general can be expressed in terms of a regular wave, a wave source, a dipole and wave-free potentials (Ursell (1968), Athanassonlis (1984)).The wave-free potentials are singular at some point and tend to zero rapidly at infinity. Obviously these satisfy the free-surface condition. Expansions in terms of the wave source and an infinite set of wave-free potentials were introduced for the three-dimensional problem involving a floating sphere half-immersed and making periodic heaving oscillations by Havelock (1955). Two and three-dimensional multipole expansions in the theory of surface waves in infinitely deep water and also in water of uniform finite depth has been given by Thorne (1953). Rhodes-Robinson (1970) constructed wave-free potentials in the presence of surface tension at the free surface. Mandal and Goswami (1984) studied oblique scattering by a half-immersed circular cylinder by using two methods, one based on integral equation formulation and other based on expansion of the scattered velocity potential by the method of multipoles.

Thus for various classes of water wave problems many researchers use the wave-free potentials in the mathematical analysis. In most of these works the expressions of wave free potentials are only given without their method of derivation. However, Linton and McIver (2001) indicated briefly how these can be constructed in case of water with a free surface.

Recently there is a considerable interest in the mathematical investigation of ice-wave interaction problems due to an increase in the scientific activities in polar oceans. Instead of a free surface, a polar ocean in covered by ice. The ice cover is modelled as a thin uniform sheet of ice of which still a smaller part is immersed in water, and is composed of materials having elastic properties. Already, quite a number of researchers have considered various types of water wave problems in a polar ocean with an ice-cover modelled as a thin elastic plate. Das and Mandal (2009) investigated wave scattering by a circular cylinder half-immersed in water with an ice-cover. They employed the method of multipoles by using the general expansion theorem for the wave potential involving wave-free potentials whose expressions were only given. Recently Das and Mandal (2010) investigated construction of wave-free potential in the linearized theory of water waves. The method of constructing these wave-free potentials was presented there in a systematic manner for a number of situations such as deep water with a free surface, neglecting or taking into account the effect of surface tension, or with an ice-cover modelled as a thin elastic plate floating on water.

In these cases then higher-order boundary conditions involves third order partial derivative (surface tension) and fifth-order partial derivatives (ice-cover) were presented. However, the boundary value problem involving higher-order boundary conditions more than fifth order partial derivatives (Manam et. al. (2006), Das et. al. (2008), Das (2015)) have not been extensively studied with a view to establish the wave-free potentials in a single layer fluid.

In this paper, we extend the problems of Das and Mandal (2010), Dhillon and Mandal (2014) investigated the problem of wave-free potentials in water wave theory for free surface boundary condition with higher-order derivatives and presented in a systematic manner. When the higher-order partial derivative reduces to 1st order (free surface) or, third order (surface tension) or fifth order (ice-cover), wave free potentials exactly coincide with the wave free potentials for two dimension (cf. Das and Mandal (2010)) and also for three dimension (cf. Dhillon and Mandal (2014)).

II. Formulation Of The Problem

The usual assumptions of incompressible, homogeneous and inviscid fluid, irrotational and simple harmonic motion with angular frequency ω under gravity only, are made. A rectangular cartesian co-ordinate system is chosen with its origin on the mean horizontal position of the upper surface of the fluid taken as (x,z) plane and y-axis is taken to be vertically downwards into the fluid region. We first consider solutions of Laplace equation which are singular at $(0, f>0)$. Let θ,θ' be the angles defined by

$$\tan\theta = \frac{x}{y-f}, \quad \tan\theta' = -\frac{x}{y+f}$$

and let $r,\ r'$ denote the radial distances of the point (x,y) from the points $(0,f)$ and $(0,-f)$ $(f>0)$ respectively.

2.1 Non Oblique wave

If $Re\{\phi(x,y)e^{-iwt}\}$ denotes the velocity potential describing the motion in the fluid, the $\phi(x,y)$ satisfies

$$\nabla^2\phi = 0, \qquad \text{in the fluid region,} \tag{2.1}$$

where ∇^2 denotes the two-dimensional Laplace operator.
The bottom condition is given by

$$\nabla\phi(x,y) \to \infty \quad as \quad y\to\infty \tag{2.2}$$

Also $\phi(x,y)$ behaves as outgoing waves as $|x|\to\infty$.

The potential function $\phi(x,y)$ satisfies (2.1), (2.2) and also linearized boundary condition for higher-order derivatives has been introduced by Manam et. al.(2006) and has the form

$$\mathcal{L}\phi_y + K\phi = 0 \quad on \quad y=0, \tag{2.3}$$

where $\mathcal{L}$ is a linear differential operator of the form

$$\mathcal{L} = \sum_{m=0}^{m_0} C_m \frac{\partial^{2m}}{\partial x^{2m}}. \tag{2.4}$$

In (2.4) C_m $(m=0,1,...,m_0)$ are known constants. Keeping in mind various physical problems involving fluid structure interaction, only the even order partial derivatives in x are considered in the differential operator $\mathcal{L}$.

Let ϕ_n^s and ϕ_n^a denote the symmetric and anti-symmetric multipoles satisfying (2.1) except at $(0,f)$ with boundary conditions (2.2), (2.3) and

$$\phi_n^s \to \frac{\cos n\theta}{r^n} \qquad as \quad r \to 0, \tag{2.5}$$

$$\phi_n^a \to \frac{\sin n\theta}{r^n} \qquad as \quad r \to 0, \tag{2.6}$$

Also they represent outgoing waves as $|x| \to \infty$.The suitable multipoles are

$$\phi_n^s = \frac{\cos n\theta}{r^n} + \int_0^\infty A(\mathrm{k})\,\mathrm{e}^{-ky}\cos kx\,dk, \tag{2.7}$$

$$\phi_n^a = \frac{\sin n\theta}{r^n} + \int_0^\infty B(k)\,\mathrm{e}^{-ky}\sin kx\,dk, \tag{2.8}$$

where $A(k)$ and $B(k)$ are functions of k to be found such that the integrals exist in some sense and boundary condition (2.3) is satisfied. The unknown constants are obtained as (cf. Das and Mandal (2010))

$$A(k) = \frac{(-1)^n}{(\mathrm{n}-1)!}k^{n-1}\frac{(\sum_{m=0}^{m_0}(-1)^m C_m k^{2m+1} + K)}{H(k)}e^{-kf} \tag{2.9}$$

$$B(k) = \frac{(-1)^{n+1}}{(\mathrm{n}-1)!}k^{n-1}\frac{(\sum_{m=0}^{m_0}(-1)^m C_m k^{2m+1} + K)}{H(k)}e^{-kf} \tag{2.10}$$

where

$$H(k) = \sum_{m=0}^{m_0}(-1)^m C_m k^{2m+1} - K.$$

Thus we have

$$\phi_n^s = \frac{\cos n\theta}{r^n} + \frac{(-1)^n}{(\mathrm{n}-1)!}\int_0^\infty k^{n-1}\frac{(\sum_{m=0}^{m_0}(-1)^m C_m k^{2m+1} + K)}{H(k)}e^{-k(y+f)}\cos kx\,dk, \tag{2.11}$$

$$\phi_n^a = \frac{\sin n\theta}{r^n} + \frac{(-1)^{n+1}}{(\mathrm{n}-1)!}\int_0^\infty k^{n-1}\frac{(\sum_{m=0}^{m_0}(-1)^m C_m k^{2m+1} + K)}{H(k)}e^{-k(y+f)}\sin kx\,dk, \tag{2.12}$$

where the contour of the integrals is indented below the pole $k = \lambda$ on the real k -axis, λ being the only real positive root of the dispersion equation

$$H(k) = \sum_{m=0}^{m_0}(-1)^m C_m k^{2m+1} - K = 0. \tag{2.13}$$

The far-field forms of the multipoles are given by

$$\phi_n^s \sim \frac{2K}{H'(\lambda)}\pi i a_n e^{-\lambda(y+f)}e^{i\lambda|x|}, \tag{2.14}$$

$$\phi_n^a \sim \frac{2K}{H'(\lambda)} \pi i b_n e^{-\lambda(y+f)} e^{i\lambda|x|}, \tag{2.15}$$

as $|x| \to \infty$ where

$$a_n = \frac{(-1)^n}{(\text{n}-1)!} \lambda^{n-1}, \tag{2.16}$$

$$b_n = \frac{(-1)^{n+1}}{(\text{n}-1)!} \lambda^{n-1}. \tag{2.17}$$

Using (2.16) and (2.17), we find

$$a_n + \frac{\lambda}{(\text{n}-1)} a_{n-1} = 0,\ b_n + \frac{\lambda}{(\text{n}-1)} b_{n-1} = 0. \tag{2.18}$$

Thus,

$$\phi_n^s + \frac{\lambda}{(\text{n}-1)} \phi_{n-1}^s = \frac{\cos n\theta}{r^n} + \frac{\lambda}{(\text{n}-1)} \frac{\cos(n-1)\theta}{r^{n-1}} + \frac{(-1)^n}{(\text{n}-1)!} \int_0^\infty k^{n-2} g_1(k) e^{-k(y+f)} \cos kx dk, \tag{2.19}$$

$$\phi_n^a + \frac{\lambda}{(\text{n}-1)} \phi_{n-1}^a = \frac{\sin n\theta}{r^n} + \frac{\lambda}{(\text{n}-1)} \frac{\sin(n-1)\theta}{r^{n-1}} + \frac{(-1)^{n+1}}{(\text{n}-1)!} \int_0^\infty k^{n-2} g_1(k) e^{-k(y+f)} \sin kx dk, \tag{2.20}$$

where

$$\text{g}_1(k) = \frac{\left(\sum_{m=0}^{m_0} (-1)^m C_m k^{2m+1} + K\right)}{\text{g}(k)} \tag{2.21}$$

$$g(k) = (-1)^{m_0} c_{m_0} (k^{2m_0} + \lambda k^{2m_0-1} + \lambda^2 k^{2m_0-2} + ... + \lambda^{2m_0}) + (-1)^{m_0-1} c_{m_0-1} (k^{2m_0-1} + \lambda k^{2m_0-2} + \lambda^2 k^{2m_0-3} + ... + \lambda^{2m_0-1}) + ... - \text{c}_1(k+\lambda) + \text{c}_0.$$

Making $f \to 0$, then from (2.19) and (2.20) we get symmetric and anti symmetric wave-free potentials are given by

$$\chi_m^{(s)} = \frac{\cos \text{m}\theta}{r^m} + \frac{\lambda}{(\text{m}-1)} \frac{\cos(\text{m}-1)\theta}{r^{m-1}} + \frac{(-1)^m}{(m-1)!} \int_0^\infty k^{m-2} g_1(k) e^{-ky} \cos kx dk, \quad m = 1, 2, 3, ..., \tag{2.22}$$

and

$$\chi_m^{(o)} = \frac{\sin m\theta}{r^m} + \frac{\lambda}{(m-1)} \frac{\sin(m-1)\theta}{r^{m-1}}$$

$$+ \frac{(-1)^{m+1}}{(m-1)!} \int_0^\infty k^{m-2} g_1(k) e^{-ky} \sin kx dk, \quad m = 1, 2, 3, ... \tag{2.23}$$

In particular, choose $c_0 = 1, c_i = 0, i = 1, 2, ..., m_0$, the boundary value problem (BVP) becomes the BVP for water with free surface (cf. Das and Mandal (2010)) and the wave-free potentials become the wave-free potentials for single layer fluid with free surface. Similarly, if choose $c_0 = 1 - \in K, c_1 = 0, c_2 = D,$ $c_i = 0, i = 3, 4, 5, ..., m_0$, then the BVP becomes the BVP for fluid with ice cover boundary condition and obtain wave-free potentials (cf. Das and Mandal (2010)).

2.2 Oblique wave

Under the usual assumptions of linear water wave theory a velocity potential can be defined for oblique waves in the form

$$\Phi(x, y, z, t) = Re\{\phi(x, y) e^{-i\sigma t + i\gamma z}\}$$

where $\phi(x, y)$ is a complex valued potential function, γ is the wave number component along the z-direction. ϕ satisfies Helmholtz equation

$$(\nabla^2 - \gamma^2)\phi = 0, \qquad \text{in the fluid region.} \tag{2.24}$$

On the upper surface having the mean position $y = 0, \phi$ satisfies the free-surface condition with higher-order derivatives of the form (cf. Manam et al (2006))

$$\mathcal{M}\phi_y + K\phi = 0 \qquad on \quad y = 0, \tag{2.25}$$

where $\mathcal{M}$ is a linear differential operator of the form

$$\mathcal{M} = \sum_{m=0}^{m_0} c_m \left(\frac{\partial^2}{\partial x^2} - \gamma^2 \right)^m . \tag{2.26}$$

In (2.26) $c_m (m = 0, 1, ..., m_0)$ are known constants. Keeping in mind various physical problems involving fluid structure interaction, only the even order partial derivatives in x are considered in the differential operator $\mathcal{M}$.

Let ϕ_n^s and ϕ_n^a denote the symmetric and anti-symmetric multipoles satisfying (2.24) except at $(0, f)$ with boundary conditions (2.25), (2.2) and

$$\phi_n^s \to K_n(\gamma r) \cos n\theta \qquad as \quad r \to 0, \tag{2.27}$$

$$\phi_n^a \to K_n(\gamma r)\sin n\theta \qquad as \quad r \to 0 \tag{2.28}$$

where $K_n(z)$ denotes the modified Bessel function of second kind.
The multipoles are constructed as (cf. Thorne (1953))

$$\phi_n^s = K_n(\gamma r)\cos n\theta + \int_0^\infty A_1(k)\cos(\gamma x\sinh k)e^{-\gamma y\cosh k}dk, \tag{2.29}$$

$$\phi_n^a = K_n(\gamma r)\sin n\theta + \int_0^\infty B_1(k)\sin(\gamma x\sinh k)e^{-\gamma y\cosh k}dk \tag{2.30}$$

where $A_1(k)$ and $B_1(k)$ are functions of k to be obtained such that the integrals exist in some sense and the boundary condition (2.25) is satisfied.
The surface condition (2.25) is satisfied if $A_1(k)$ and $B_1(k)$ are chosen as

$$A_1(k) = (-1)^n \cosh nk \frac{\upsilon(\sum_{m=0}^{m_0}(-1)^m c_m \upsilon^{2m}) + K}{H(\upsilon)} e^{-\upsilon f}, \tag{2.31}$$

$$B_1(k) = (-1)^n \sinh nk \frac{\upsilon(\sum_{m=0}^{m_0}(-1)^m c_m \upsilon^{2m}) + K}{H(\upsilon)} e^{-\upsilon f}, \tag{2.32}$$

where

$$\upsilon = \gamma\cosh k.$$

Thus we can construct the multipoles are given by

$$\phi_n^s = K_n(\gamma r)\cos n\theta$$
$$+(-1)^n \int_0^\infty \frac{\upsilon(\sum_{m=0}^{m_0}(-1)^m C_m \upsilon^{2m}) + K}{H(\upsilon)} \cosh nk\cos(\gamma x\sinh k)e^{-\upsilon(y+f)}dk \tag{2.33}$$

$$\phi_n^a = K_n(\gamma r)\sin n\theta$$
$$+(-1)^n \int_0^\infty \frac{\upsilon(\sum_{m=0}^{m_0}(-1)^m C_m \upsilon^{2m}) + K}{H(\upsilon)} \sinh nk\sin(\gamma x\sinh k)e^{-\upsilon(y+f)}dk \tag{2.34}$$

where the contour is indented below the pole $k = \mu$ on the real k -axis to take care of the outgoing nature as $|x| \to \infty$, where

$$\gamma\cosh\mu = \lambda.$$

where λ being the only real positive root of the dispersion equation

$$H(\upsilon) = \upsilon(\sum_{m=0}^{m_0}(-1)^m c_m \upsilon^{2m}) - K = 0. \tag{2.35}$$

The far-field forms of the multipoles are given by

$$\phi_n^s \sim 2\pi i a_n^{(1)} \frac{K}{H'(\lambda)\lambda\cos\alpha} e^{-\lambda(y+f)} e^{i\lambda|x|\cos\alpha}, \tag{2.36}$$

$$\phi_n^a \sim 2\pi i b_n^{(1)} \frac{K}{H'(\lambda)\lambda\cos\alpha} e^{-\lambda(y+f)} e^{i\lambda|x|\cos\alpha} \tag{2.37}$$

as $|x| \to \infty$, where

$$a_n^{(1)} = (-1)^n \cosh n\mu_2, \tag{2.38}$$

$$b_n^{(1)} = (-1)^n \sinh n\mu_2. \tag{2.39}$$

Using (2.38) and (2.39), we find

$$a_{n-2}^{(1)} + \frac{2\lambda}{\gamma} a_{n-1}^{(1)} + a_n^{(1)} = 0, \tag{2.40}$$

$$b_{n-1}^{(1)} + \frac{2\lambda}{\gamma} b_n^{(1)} + b_{n+1}^{(1)} = 0. \tag{2.41}$$

Thus

$$\begin{aligned}\phi_{n-2}^s + \frac{2\lambda}{\gamma}\phi_{n-1}^s + \phi_n^s = K_{n-2}(\gamma r)\cos(n-2)\theta + \frac{2\lambda}{\gamma} K_{n-1}(\gamma r)\cos(n-1)\theta \\ + K_n(\gamma r)\cos n\theta \\ + \frac{2(-1)^n}{\gamma}\int_0^\infty g_1(\upsilon)\cosh(n-1)k\cos(\gamma x\sinh k)e^{-\upsilon(y+f)}dk\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}\phi_{n-1}^a + \frac{2\lambda}{\gamma}\phi_n^a + \phi_{n+1}^a = K_{n-1}(\gamma r)\sin(n-1)\theta + \frac{2\lambda}{\gamma} K_n(\gamma r)\sin n\theta \\ + K_{n+1}(\gamma r)\sin(n+1)\theta \\ + \frac{2(-1)^{n-1}}{\gamma}\int_0^\infty g_1(\upsilon)\sinh nk\sin(\gamma x\sinh k)e^{-\upsilon(y+f)}dk.\end{aligned} \tag{2.43}$$

These are wave-free potentials with singularity at $(0, f)$.

Making $f \to 0$ in (2.42) and (2.43) we obtain the symmetric and anti-symmetric wave-free potentials with singularity in the free surface and are given by

$$\chi_m^s = K_{m-2}(\gamma r)\cos(m-2)\theta + \frac{2\lambda}{\gamma}K_{m-1}(\gamma r)\cos(m-1)\theta + K_m(\gamma r)\cos m\theta$$

$$+\frac{2(-1)^m}{\gamma}\int_0^\infty g_1(\upsilon)\cosh(m-1)k\cos(\gamma x\sinh k)e^{-\upsilon y}dk, \quad m=1,2,3,...$$

(2.44)

and

$$\chi_m^a = K_{m-1}(\gamma r)\sin(m-1)\theta + \frac{2\lambda}{\gamma}K_m(\gamma r)\sin m\theta + K_{m+1}(\gamma r)\sin(m+1)\theta$$

$$+\frac{2(-1)^{m-1}}{\gamma}\int_0^\infty g_1(\upsilon)\sinh mk\sin(\gamma x\sinh k)e^{-\upsilon y}dk, \quad m=1,2,3,...$$

(2.45)

These have been used by Das and Mandal (2009) in the study of wave scattering by a long circular cylinder half-immersed in water with an ice-cover. Here also in particular, the results for water with free surface as well as ice-cover surface are similar to the case of non oblique wave potential (cf. Das and Mandal (2010)).

2.3 Three-Dimensional Wave-Free Potentials

With the origin at the mean free surface, the x and z -axes horizontal and the y -axis vertical, y increasing with depth, we define the angles θ, θ' and α by the relations

$$\tan\theta = \frac{R}{y-f}, \ \tan\theta' = -\frac{R}{y+f}, \ \tan\alpha = \frac{z}{x}$$

where $R=\sqrt{x^2+z^2}$. Let r and r' denote the radial distances of the point (x,y,z) from the points $(0,f,0)$ and $(0,-f,0)$ respectively.

If $Re\{\phi(x,y,z)e^{-iwt}\}$ denote the velocity potential singular at $(0,f,0)$ describing the motion in the fluid, then $\phi(x,y,z)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad \text{in the fluid region except at } (0,f,0). \tag{2.46}$$

The bottom condition for water of infinite depth is given by

$$\nabla\phi(x,y,z)\to 0 \quad as \quad y\to\infty. \tag{2.47}$$

Also $\phi(x,y,z)$ behaves as outgoing waves as $R\to\infty$.

The potential function $\phi(x,y,z)$ satisfies (2.46), (2.47) and the linearized condition given by

$$\left(\sum_{s=0}^{m_0}C_s\nabla_{x,z}^{2s}\right)\phi_y + K\phi = 0 \qquad on \quad y=0. \tag{2.48}$$

In (2.48) C_s $(\mathrm{s}=0,1,2,...,m_0)$ are known constants,

$$\nabla_{x,z}^{2s}=\left[\nabla_R^2+\frac{1}{R^2}\frac{\partial^2}{\partial\alpha^2}\right]^s$$

where

$$\nabla_R^2=\left[\frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial}{\partial R})\right]. \tag{2.49}$$

In this case (cf. Dhillon and Mandal (2014)),

$$\phi_n^m=\frac{p_n^m(\cos\theta)}{r^{n+1}}+\int_0^\infty A_2(\mathrm{k})\,\mathrm{e}^{-ky}J_m(kR)\,dk, \tag{2.50}$$

and $A_2(\mathrm{k})$ is a function of k to be obtained such that the integral exists and the boundary condition (2.48) is satisfied. The condition (2.48) is satisfied if $A_2(\mathrm{k})$ is chosen as

$$A_2(k)=\frac{(-1)^n}{(n-m)!}\frac{\left[\sum_{s=0}^{m_0}(-1)^s C_s k^{2s+1}+K\right]}{H(k)}k^n e^{-kf}, \tag{2.51}$$

Thus we get

$$\phi_n^m=\frac{p_n^m(\cos\theta)}{r^{n+1}}+\frac{(-1)^n}{(n-m)!}\int_0^\infty\frac{\left[\sum_{s=0}^{m_0}(-1)^s C_s k^{2s+1}+K\right]}{H(k)}k^n\,\mathrm{e}^{-k(y+f)}J_m(kR)\,dk, \tag{2.52}$$

where the contour of the integral is defined below the pole $k=\lambda$ on the real k axis, λ being the only real positive root of the dispersion equation

$$H(k)=\sum_{s=0}^{m_0}(-1)^s C_s k^{2s+1}-K=0. \tag{2.53}$$

The far-field form of the multipole is given by

$$\phi_n^m\sim\frac{4K}{H'(\lambda)}\pi i a_n^{(2)}g(\lambda)e^{-\lambda(y+f)}\sqrt{\frac{2}{\pi\lambda R}}e^{i(\lambda R-\frac{\pi}{4})}\quad as\quad R\to\infty,$$

where

$$a_n^{(2)}=\frac{(-1)^n}{(n-m)!}\lambda^n e^{-im\frac{\pi}{2}}. \tag{2.54}$$

From (2.54) we get,

$$a_{n+1}^{(2)} + \frac{\lambda}{(n-m+1)} a_n^{(2)} = 0. \tag{2.55}$$

Therefore, the combination $\phi_{n+1}^m + \frac{\lambda}{(n-m+1)}\phi_n^m$ does not contribute anything as $R \to \infty$, so that they are wave-free. Thus,

$$\psi_n^m = \phi_{n+1}^m + \frac{\lambda}{(n-m+1)}\phi_n^m = \frac{p_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\lambda}{(n-m+1)}\frac{p_n^m(\cos\theta)}{r^{n+1}}$$

$$+\frac{(-1)^{n+1}}{(n-m+1)!}\int_0^\infty g_1(k)k^n\, \mathrm{e}^{-k(y+f)} J_m(kR)\,dk, \tag{2.56}$$

This is the wave-free potential having singularity at $(0, f, 0)$. Making $f \to 0$ in (2.56) we find the wave-free potential having singularity in the free surface and is given by

$$\chi_n^m = \frac{p_{n+1}^m(\cos\theta)}{r^{n+2}} + \frac{\lambda}{(n-m+1)}\frac{p_n^m(\cos\theta)}{r^{n+1}}$$

$$+\frac{(-1)^{n+1}}{(n-m+1)!}\int_0^\infty g_1(k)k^n\, \mathrm{e}^{-ky} J_m(kR)\,dk. \tag{2.57}$$

In particular, choose $c_0 = 1, c_i = 0, i = 1, 2, ..., m_0,$ the boundary value problem (BVP) becomes the BVP for water with free surface (cf. Dhillon and Mandal (2014)) and the wave-free potentials become the wave-free potentials for single layer fluid with free surface. Similarly, if choose $c_0 = 1 - \in K, c_1 = 0, c_2 = D,$ $c_i = 0, i = 3, 4, 5, ..., m_0$, then the BVP becomes the BVP for fluid with ice-cover boundary condition and obtain wave-free potentials (cf. Dhillon and Mandal (2014)).

III. Conclusion

Wave free potentials in single-layer fluid with a free surface condition with higher order derivatives for non-oblique and oblique waves (two dimensions) and also three-dimension are constructed in a symmetric manner. Appropriate modifications of the wave-free potentials can be made in the circumstances when the fluid are of uniformly finite depth having a free surface conditions with higher order derivatives. In particular, these are obtained taking into account of the effect of the presence of free surface, surface tension at the free surface and also in the presence of an ice-cover modelled as thin elastic plate.

Acknowledgements

The authors thank the Editor and the Reviewers for their comments and suggestions to revise the paper.

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International Journal of Engineering Science Invention (IJESI) is UGC approved Journal with Sl. No. 3822, Journal no. 43302.

Nityananda Thakur and Dilip Das "Wave Free Potentials In The Theory Of Water Waves Having Free Surface Boundary Condition With Higher Order Derivatives" International Journal of Engineering Science Invention (IJESI), vol. 07, no. 01, 2018, pp. 33–43.