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A PERFORMANCE STUDY OF SHEWHART-TYPE t_r-CHARTS FOR MONITORING HIGH-YIELD PROCESSES

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Abstract:*The times between events (TBE) control chartsplay a vital role in monitoring high-yield processes where the defect rate is too low. It is well known that in monitoring high-yield processes, the traditional control charts, for example, c-chart, u-chart are found inefficient, hence, TBE charts are recommended as an alternative of these traditional charts. In this paper, we consider the Shewhart-type t_r-chart because it is most prevalent chart for monitoring TBEs and easy to implement. Usually, the average run length is used to evaluate the chart's performance, however, there are several criteria to examine the performance of the charts, such as ARL, MRL, EQL etc. In this paper, we evaluate the t_r-chart in terms of these performance criteria and recommend the appropriate chart accordingly. Note that the t_r-chart possesses the undesirable ARL-biasedness property which implies a delay in detecting a signal even the process is OOC. To overcome the drawback of biasedness, the ARL-unbiased charts are proposed. We also evaluate the performance of the ARL-unbiased t_r-chart and comparison is made between ARL-biased and unbiased t_r-charts based on the various performance criteria.*

Keywords:*Average run length (ARL), Extra quadratic loss (EQL), Median run length (MRL), Performance comparison index (PCI), Relative average run length (RARL), t_r-charts.*

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I. Introduction

Statistical Process Control (SPC) is an industry-standard methodology for measuring and controlling quality during the manufacturing process. There is a certain amount of variability in every product which is due to mainly two causes, namely common causes (chance causes) and assignable causes. An important goal of SPC is to distinguish between two sources of variation in the process.When the process operates under purely chance causes, it is said to be in state of statistical control (IC), otherwise process is out-of-control (OOC). To this end,control chart helps users to identify assignable causes so that the state ofstatistical control can be achieved [1]. If assignable causesarepresent in the process, a control chart should detect it quickly as possible and give anOOC signal. The control charts have quicker detection ability of an OOC signal, are considered as moreefficient charts. There is a widerange of control charts available in the quality control literature such Shewhart type $\bar{X}$-chart, np-chart, p-chart, c-chart, time between event (TBE)control chart etc.

There are many situations where the defects/failure rate is low, say parts per million (ppm), especially, in high-yield processes. In these situations, the conventional charts which are based on the number of failures/defects/non-conforming items, for example, p-chart,c-chartsare found to be inefficient in high-yield processes monitoring. Hence, instead of monitoring the number or the proportionof events occurring in sampling intervals, the monitoring of times between two successive failures or non-conforming items are recommended in the SPC literature [2]. The control charts which are used to monitor the TBEs are termed as TBE control charts. An advantage of TBE charts is that it makes the online monitoring much easier since it leaves the process continuous runningwithout interruption until there is an out-of-control signal occurs [3]. Moreover, it does not require a rational subgroup of suitable size, whichsometimes adversely affects the performance of the charts. An important issue regarding any control chart is its performance. There are a number of measures in literature which are suggested to evaluate the performance of control charts such as average run length (ARL), median runlength (MRL), standard deviation run length (SDRL), relative averagerun length (RARL), extra quadratic loss (EQL) and performance comparison index (PCI) etc.Several TBE control charts are recommended to monitor the high-yield processes which includes Shewhart-type exponential chart, t_r-charts, cumulative sum (CUSUM) and exponentially weighted moving average (EWMA)(see, [4], [5], [6], [7], [8],[9], [10], [11],[12],[13]).

In this paper, we consider the Shewhart-type t_r-chart because it is most prevalent and easy to implement among all the TBE charts.It is well established that the Shewhart-type t_r(also known as Erlang chart) charts are ARL-biased which is considered as a drawback of the chart because,on average, the ARL-biased chart takes a longer time to detect a signal even the process is OOC [14].This study intends to investigate the performance of ARL-biased and -unbiased Shewhart-typet_r-charts and comparestheir performances in terms of not only ARL which is most common performance criterion of evaluation of charts but also in other performance criteria such as EQL, RARL etc. These criteria might be helpful toexplore the performance of the charts in different perspective, for example, the ARL gives an idea about the performance at a specific shift in the process parameter whereas sometimes, it is of interest to know the overall performance of the chart in an interval of shift. The criteria which examine the chart's performance over a shift interval include EQL, RARL, PCI(see, [15], [16], [17], [18], [19], [20], [21], [22] and [23]).

Rest of the paper is organized as follows. The control limits of the Shewhart-type ARL-biased and unbiasedt_r-chart are designed in Section 2. In Section 3, the various performance criteria are discussed. Next, the performance oft_r-charts are examined in terms of various performance measures in Section 4. Based on the performance study, the recommendations are made in Section 5.

II. Control limits of the ARL-biased and –unbiased t_r-charts

In high-yield processes where the failure rate is low, the occurrences of failures can be modeled by ahomogeneous Poissonprocess [24].It is well known that for a Poisson process with constant rate parameter, say λ, theinter-arrival times (times between two failures or non-conforming items) are independent and identically distributed and followexponential distribution with mean $1/\lambda$. Let X denotes the TBE random variable,which follows exponential distribution with probability densityfunction (pdf)given by

$$f(x) = \lambda \exp(-\lambda x) \ ; x > 0; \lambda > 0 \tag{1}$$

where, λ is the failure rate or rate parameter. Let us denote the known value or specified value of λ by λ_0. The lower (LCL) and upper (UCL) control limits for t-chart or exponential chart, proposed by [5] in terms of percentile of chi-square distribution and following equal tail probability approach are given by (see also, [24])

$$LCL = \frac{\chi^2_{2,\frac{\alpha}{2}}}{2\lambda_0} \quad \text{and} UCL = \frac{\chi^2_{2,1-\frac{\alpha}{2}}}{2\lambda_0} \tag{2}$$

Center line for the t-chart,taken as median of distribution of$X$, is given by;

$$CL = \frac{\chi^2_{2,0.5}}{2\lambda_0} \tag{3}$$

Generalization of$t$-chart has also been proposed by [5] to increase the sensitivity of control chart to detect an OOC signal by considering the charting statistic as the waiting time up to r^{th} failure, hence the chart is known ast_r-chart. Let $T_r (= \sum_{i=1}^{r} X_i \ ; r \geq 1)$ be the sum of$r$ consecutive failure times which follows a gamma (Erlang distribution) distribution with shape parameter$r$and scale parameterλ. Due to this reason, the t_r-chart is also known as Erlang-chart. The probability density function (pdf) of T_r is given as follows;

$$f(t; r, \lambda) = \frac{\lambda^r}{\Gamma(r)} t^{r-1} exp(-\lambda t) \ \ ; t > 0; \lambda > 0; r \in \{1,2, \dots\} \tag{4}$$

The control limits for t_r-chartare given by [24];

$$LCL_r = \frac{\chi^2_{2r,\frac{\alpha}{2}}}{2\lambda_0} \quad \text{and} UCL_r = \frac{\chi^2_{2r,1-\frac{\alpha}{2}}}{2\lambda_0} \tag{5}$$

Center line for the t_r-chart is taken as median of distribution ofT_r, is given by;

$$CL_r = \frac{\chi^2_{2r,0.5}}{2\lambda_0} \tag{6}$$

where,$\chi^2_{2r,\alpha}$denote 100α-percentile of chi-square distribution with $2r$ degrees of freedom and $\lambda$is the failure rate. The process is said to be in-control when $\lambda = \lambda_0$(known). It is well established in SPC literature that the charts given by (5) possess the biasedness property which implies that on an average, the charts detect a signal in delay when the process is OOC for some shift sizes than the case when the process is IC. Customarily, the biasedness property of the chart is defined in terms of most common measure of the performance, the ARL which is defined in (10) and hence, it is termed as ARL-biasedness. To visualize the ARL-biasedness property, we obtained the values of ARL function in (10) of t_r-charts ($r = 1,2,3,4$) for different rate parameters, say λ_1(which has been shifted from IC rate parameter value, sayλ_0) and FAR=0.0027. The ARL functions for each chart are depicted in Fig. 1. Moreover, the ARL function does not depend on the value of rate parameter, but it depends on the size of the shift, say $\delta = \lambda_1/\lambda_0$ where λ_0 is the IC rate parameter and$\lambda_1$is the value of the rate parameter when the process has been shifted fromλ_0 (it is shown in the following section). Therefore, without loss of generality, we can consider the shifts in the process parameter and their impact on the performance in terms of δ irrespective of the values of λ_0 andλ_1.

It can be observed from Fig. 1 that each chart has higher OOC ARL values for some $\delta > 1$ than the IC ARL value equal to 370.4 (=1/FAR). It implies that the charts take longer time to detect the out of control (OOC) signal than the in control (IC) case.To avoid this undesirable property, ARL-unbiased charts are recommendedin the literature which are defined in the following.

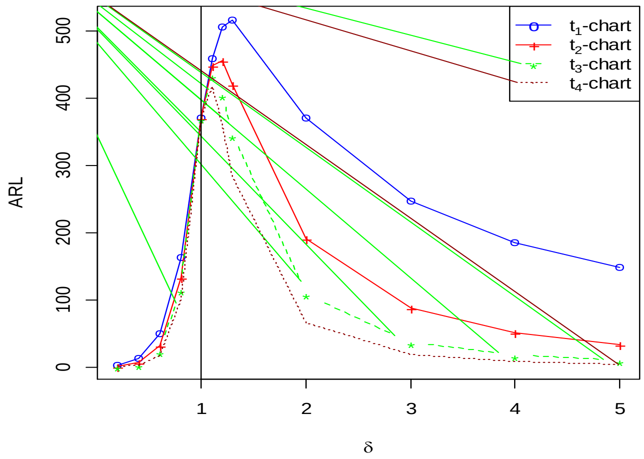

Fig.1 ARL curve of t_r-charts at$\alpha_0 = 0.0027, r\epsilon\{1,2,3,4\}$

2.1 ARL unbiased design of t_r-chart

For the ARL-unbiasedness,the Shewhart type t_r-chart is designed so that ARL function must have its maximum value when the process is in control i.e. $\lambda = \lambda_0$. Also, we set the maximum value of ARL equal tothe specified value of ARL, say ARL_0. To obtain the ARL unbiasedt_r-chart the following two conditions must be satisfied:

$$ARL(\delta) = ARL_0 \text{when} \delta = 1 \quad (7a)$$

$$ARL(\delta) < ARL_0 \text{when } \delta \neq 1 \quad (7b)$$

Now to obtain the control limits for ARL-unbiased t_r-chart we introduced two constants $\gamma$and$p$ to obtain a unique pair to satisfy (7a) and (7b). The new control limits, denoted by $LCL_{ru}$and UCL_{ru} of the ARL-unbiased t_r-chart can be obtained as follows.

$$LCL_{ru} = \frac{\chi^2_{2r,\gamma p}}{2\lambda} \quad \text{and} \quad UCL_{ru} = \frac{\chi^2_{2r,1-p}}{2\lambda} \quad (8)$$

To obtain the pair of control limit (LCL_{ru}, UCL_{ru})in (8), we need to find a pair of(γ, p)such that the ARL function must satisfy two conditions given in (7a) and (7b). After obtaining the constants (γ, p) by following the steps given in [25], we can obtain control limits of the ARL-unbiased t_r-charts in (8). The ARL function of unbiased t_r-charts is given in (11) which is shown in Fig. 2 for t_r-chart ($r \in \{1,2,3,4\}$).For more details regarding the ARL-unbiased charts, the readers are referred to [25].

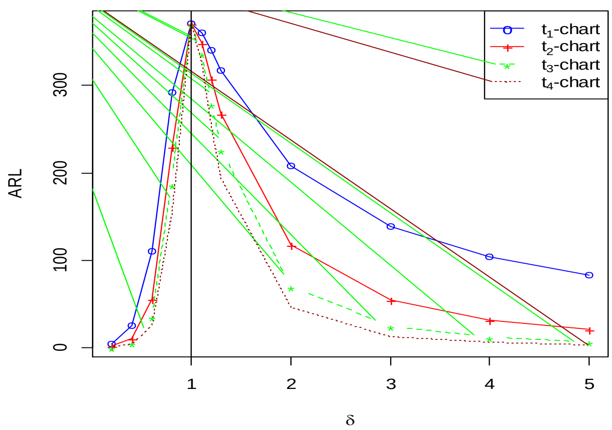

Fig.2 ARL curve of unbiasedt_r-charts at$\alpha_0 = 0.0027, r\epsilon\{1,2,3,4\}$

From the above Fig. 2, it is seen that for each t_r-chart ($r = 1,2,3,4$), the IC (i.e. nominal) value of ARL is higher than the OOC ARL values for each $\delta \neq 1$which implies that ARL-unbiased t_r-charts take less time to detect OOC situation than the IC case, which is a desirable characteristic of a good control chart.

Next, we evaluate the performance of both ARL-biased and -unbiased t_r-charts in terms of various performance criteria which are discussed below.

III. Performance Measures

Once the control limits of the charts are constructed, it is of interest to evaluate their performance. There are several criteria to evaluate the control chart's performance which can be classified according to criteria which evaluate the performance at a specific shift in the process parameter and/or over an interval of shift [26].Theperformance criteria which evaluate the chart at a fixed shift size include the ARL and MRL. The other class includes the measures EQL, RARL, PCI which evaluate the chart's performance over an interval of shift. Note that all the measures eventually depend on the run length (RL) distribution or in other words, the probability of signal. The probability of signal, denoted by β is defined as the probability of charting points falling outside the control limits. Note that RL follows a geometricdistribution for the Shewhart-type control chart with parameter β which is the probability of signal.

Suppose the IC parameter λ has been shifted from $\lambda_0$to λ_1 so that $\delta = \lambda_1/\lambda_0$ where δ denotes the size of the shift in IC parameter valueλ_0. Using the transformation, $2\lambda_1 T_r \sim \chi^2_{2r}$, the probability of signal of the t_r-chart with the control limits in (5) is given by

$$\begin{aligned}\beta = P[\text{signal}] &= P[T_r < LCL_r \text{or} T_r > UCL_r] \\ &= 1 - P[LCL_r \leq T_r \leq UCL_r] \\ &= 1 + F_{\chi^2_{2r}}(2\lambda_1 LCL_r) - F_{\chi^2_{2r}}(2\lambda_1 UCL_r) \\ &= 1 + F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,\frac{\alpha_0}{2}}\right) - F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,1-\frac{\alpha_0}{2}}\right)\end{aligned} \quad (9)$$

where, $F_{\chi^2_{2r}}(.)$ is the cumulative distribution function (c.d.f.) of chi-square distribution with $2r$d.f.. When $\delta = 1$ i.e. when the process is IC and the probability of signal β gives the false alarm rate (i.e. FAR). Note that β depends on δ and α_0, not on λ_0, thereafter we will denote it as $\beta(\delta, \alpha_0)$.

3.1 Performance measures for specific shift

3.1.1 Average run length (ARL)

Average run length (ARL) is the most popular metric to evaluate the performance of the control charts. ARL is the expected value of the run length distribution. Recall that the run length is the number of plotted points on the control chart until the chart gives a signal and hence, the ARL is defined as the average number of plotting points to get an OOC signal. Note that for the Shewhart-type charts, the run length follows a geometric distribution with parameter $\beta(\delta, \alpha_0)$ and thus, the ARLis the reciprocal of the probability of signal.Therefore, for given values of r and α_0 the ARL function for ARL-biased t_r-chart is given by

$$ARL(\delta, \alpha_0) = \frac{1}{1+F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,\frac{\alpha_0}{2}}\right)-F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,1-\frac{\alpha_0}{2}}\right)} \quad (10)$$

In like manner, the ARL function of ARL-unbiased t_r-charts is given by;

$$ARL(\delta, \alpha_0) = \frac{1}{1+F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,\gamma p}\right)-F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,1-p}\right)} \quad (11)$$

The IC ARL value can obtained from (10) and (11) by letting $\delta = 1$ for the ARL-biased and -unbiased charts respectively whereas for the values of$\delta \neq 1$, the ARL functions give the OOC ARL values. Clearly, smaller values of OOC ARL are desirable in the sense the chart is able to detect OOC signal quickly. Note that ARL function in (11) also depends on α_0 because the constants (γ, p) depend on FAR, α_0 for given$r$.

3.1.2 Median Run Length

Recall that the run length distribution is geometric which is a right skewed distribution. Recently, the ARL as a representative of run length distribution is under criticism in SPC literature because average is not a typical value of the skewed distribution. Hence, as an alternative of ARL, the median run length (MRL) is recommended to have a true picture about the run length distribution and performance of the control chart [1].The median run length, denoted by MRL is defined as follows.

$$P[R > MRL] < \frac{1}{2} \text{and} P[R \geq MRL] \geq \frac{1}{2} \quad (12)$$

where, R denotes RL variable following geometric distribution with parameter $\beta(\delta, \alpha_0)$. Like ARL, the MRL is also a function of δ and α_0.The equation (12) can also be re-expressed as following.

$$1-2^{-\frac{1}{MRL}} < 1+F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,\frac{\alpha_0}{2}}\right)-F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,1-\frac{\alpha_0}{2}}\right) \le 1-2^{\frac{-1}{MRL-1}} \quad (13)$$

for the MRL-biased t_r-chart and

$$1-2^{-\frac{1}{MRL}} < 1+F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,\gamma p}\right)-F_{\chi^2_{2r}}\left(\delta\chi^2_{2r,1-p}\right) \le 1-2^{\frac{-1}{MRL-1}} \quad (14)$$

for the MRL-unbiased t_r-chart.

3.2 Performance Measures for a shift interval

Sometimes, it is of interest to evaluate the chart's performance in an interval of shift size rather than at a single point of shift. There are several criteria which are recommended in the literature to examine the performance of the chart over an interval of the shift such as EQL, RARL and PCI which are described below.

3.2.1 Extra Quadratic Loss Function

The extra quadratic loss function (EQL) is defined as the weighted average of ARL function over the range ($\delta_{min}, \delta_{max}$) of process shift ($\delta$), assuming that$\delta$ varies uniformly over the interval. The weights are taken as square of shift, (i.e.δ^2) (see, [15]). A low value of EQL is desirable for an efficient chart. The EQL can be expressed as

$$EQL = \frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}} \delta^2 ARL(\delta)d\delta$$

Note that the EQL was initially proposed to evaluate the chart performance for monitoring the characteristic following normal distribution. Therefore, we use a slightly modified expression for EQL in monitoring TBEs as follows.

$$EQL = \frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}} (1-\delta)^2 ARL(\delta)d\delta \quad (15)$$

Because we are interested to measure the loss in shifting of δ from its IC value equal to 1.

3.2.2 Relative average run length

The relative average run length (RARL) measure is used to compare the overall performance of different control charts with respect to benchmark chart over a range of shift. Mathematically, we can express it as follows:

$$RARL = \frac{1}{\delta_{max}-\delta_{min}}\int_{\delta_{min}}^{\delta_{max}} \frac{ARL(\delta)}{ARL_{bmk}(\delta)} d\delta \quad (16)$$

where, δ is the shift size, ARL(δ) and ARL_{bmk} (δ)is the ARL of a chart and the benchmark chart respectively. It provides the information that how much the performance of a chart deviates from the benchmark chart. Here, we considered the chart with minimum EQL value as benchmark chart. Clearly, when the RARL > 1 (< 1), the chart is inferior (superior) to the benchmark chart in terms of performance and if $RARL = 1$, the chart performs equally to the benchmark chart.

3.2.3 Performance comparison index (PCI)

The PCI is the ratio of EQL of reference chart and the benchmark chart under the similar conditions. It provides the index for the reference chart relative to the benchmark chart over the range of shift. It enables us to give the rank to the charts under comparison based on EQL. It is defined as

$$PCI = \frac{EQL}{EQL_{bmk}} \quad (17)$$

Note that PCI $= 1$ for benchmark chart, PCI > 1 for inferior charts and < 1 for superior charts.

IVPerformance Evaluation and Comparison

In this section, the Shewhart type t_r-chart (ARL-biased and -unbiased) are evaluated and compared in terms of various performance criteria discussed in the earlier section. As stated above, the probability of signal and hence, the performance criteria do not depend on the IC rate parameter, but they depend on the size of the shift δ i. e. how much the process parameter has been shifted from λ_0 to λ_1.We first calculate the individual measures ARL and MRL and then we will obtain the overall performance measures. For individual measures, we considered the size of the shift $\delta = 5, 4, 3, 2, 1.5, 1, 0.8, 0.6, 0.4, 0.2, 0.1$ and FAR=0.0027 and obtain the ARL and MRL values of the t_r-charts ($r = 1,2,3,4$) for each δ which are reported in Tables I and II respectively.

Table I. ARL values of ARL-biased and -unbiased Shewhart type t_r-charts (r={1,2,3,4}) for $\alpha = 0.0027$

	t_1-chart		t_2-chart		t_3-chart		t_4-chart	
Shift (δ)	ARL-biased	ARL-unbiased	ARL-biased	ARL-unbiased	ARL-biased	ARL-unbiased	ARL-biased	ARL-unbiased

5	148.55	83.54	34.05	21.51	10.95	7.75	4.85	3.78
4	185.56	104.30	51.4	32.15	18.39	12.72	8.42	6.33
3	247.25	138.90	88.26	54.66	37.43	25.25	18.77	13.58
2	370.37	208.09	191.77	117.60	108.24	71.20	66.56	46.25
1.5	482.18	276.90	332.49	204.23	236.66	154.03	175.36	119.46
1	**370.37**	**370.18**	**370.37**	**370.47**	**370.37**	**370.52**	**370.37**	**370.36**
0.8	162.83	291.80	134.48	230.35	115.46	187.02	101.09	155.95
0.6	50.54	110.21	32.37	56.53	23.44	36.04	18.09	25.67
0.4	13.95	25.18	7.7	11.07	5.22	6.74	3.92	4.75
0.2	3.75	5.07	2.13	2.48	1.59	1.73	1.33	1.41
0.1	1.94	2.25	1.29	1.36	1.11	1.13	1.04	1.05

The ARL values for the ARL-biased and -unbiased t_r-charts are computed from equations (10) and (11) respectively for r=1, 2, 3 and 4 with α_0=0.0027 i.e. ARL_0=370.4. Some conclusions have been drawn from Table I which also supports the existing findings in SPCliterature. First, the ARL-unbiased t_r-charts have lower OOC ARL values than ARL-biased charts for each r so they can give a signal more quickly when the process parameter shifts from their IC value. Second, we are often more interested in case when the process deteriorates (detecting an increase inλ) from its IC status and for these cases the ARL-unbiased chart has much better performance for each$r$. Third, the ARL-unbiased t_r-chart performs better for higher values of r in terms of the ARL values. Hence, unbiased t_4–chart performs better among the considered charts. HHowever, the ARL-unbiased t_r-chart appears to be less efficient in detecting a decrease in λ(i.e. improvement case)for all $r = 1,2,3,4$.Since this is the improvement case, this doesn't seem to be a serious practical problem. Similar conclusions based on MRL values are drawn form Table II as that of ARL values.

Table II. The MRL values of MRL-biased and -unbiased Shewhart type t_r-chart (r = {1, 2, 3, 4}) for $\alpha = 0.0027$.

Shift (δ)	t_1-chart		t_2-chart		t_3-chart		t_4-chart	
	MRL-biased	MRL-unbiased	MRL-biased	MRL-unbiased	MRL-biased	MRL-unbiased	MRL-biased	MRL-unbiased
5	103	58	24	15	8	6	4	3
4	129	72	36	22	13	9	6	5
3	172	96	61	38	26	18	13	10
2	257	144	133	82	75	50	46	32
1.5	334	192	231	142	164	107	122	83
1	**257**	**257**	**257**	**257**	**257**	**257**	**257**	**257**
0.8	113	202	93	160	80	130	70	108
0.6	35	77	23	39	16	25	13	18
0.4	10	18	5	8	4	5	3	3
0.2	3	4	2	2	1	1	1	1
0.1	1	2	1	1	1	1	1	1

Apart from individual measures, the overall performance measures and comparative measures are also computed.To make more clarity about the performance of the charts in different range ofδ, we consider three intervals of the shifts: large shift interval $0.2 \leq \delta \leq 5$, moderate shift interval $0.5 \leq \delta \leq 2$ and small shift interval $0.7 \leq \delta \leq 1.25$. All the overall performance measures are obtained for these shift intervals for all the ARL-biased and -unbiased t_r-charts ($r = 1,2,3,4$) using Equations (15) -(17) for FAR=0.0027. These measures are reported in Table III. For the comparison purpose, the ARL-unbiased t_4-chart is considered as benchmark chart to compute RARL and PCI measures because it has minimum EQL value among all the charts for all three shifting intervals.

Table III. EQL, RARL and PCI values of ARL-biased and unbiased t_r-chart

Criterion	Shift (δ)	t_1-chart		t_2-chart		t_3-chart		t_4-chart	
		ARL-biased	ARL-unbiased	ARL-biased	ARL-unbiased	ARL-biased	ARL-unbiased	ARL-biased	ARL-unbiased
EQL	$0.2 \leq \delta \leq 5$	867.68	489.36	267.23	167.11	108.28	74.15	55.24	40.6

	$0.5 \leq \delta \leq 2$	95.83	57.39	58.38	37.51	38.21	26.07	26.77	19.16
	$0.7 \leq \delta \leq 1.25$	7.12	7.57	6.25	6.26	5.54	5.37	4.96	4.72
RARL	$0.2 \leq \delta \leq 5$	16.04	9.44	5.23	3.45	2.29	1.67	1.27	1.00
	$0.5 \leq \delta \leq 2$	3.26	2.54	2.15	1.71	1.54	1.27	1.18	1.00
	$0.7 \leq \delta \leq 1.25$	1.29	1.46	1.16	1.24	1.07	1.10	0.99	1.00
PCI	$0.2 \leq \delta \leq 5$	21.37	12.05	6.58	4.12	2.67	1.83	1.36	1.00
	$0.5 \leq \delta \leq 2$	5.00	3.00	3.05	1.96	1.99	1.36	1.40	1.00
	$0.7 \leq \delta \leq 1.25$	1.51	1.60	1.32	1.33	1.17	1.14	1.05	1.00

It can be observed from Table III that with an increase in r, the t_r-chart becomes superior for each shifting interval, for example, ARL-biased and -unbiased t_4-charts have EQL values equal to 55.24 and 40.6 respectively, while these values are 108.28 and 74.15 for ARL-biased and -unbiased t_3-charts. Moreover, the ARL-unbiased t_r-charts perform better than ARL-biased t_r-charts for each r and shifting interval.

For large shift interval i.e. $\delta \in [0.2, 5]$, the same pattern is observed in the performance of the charts in respect of the criteria RARL and PCI as in case of EQL i.e. both RARL and PCI values of the ARL-unbiased charts are lower than that of ARL-biased charts for all values of r. Also, the values of these measures for the t_r-charts decrease as$r$ increase. However, the different patterns are foundin the overall performance criteria over the moderate and small shift intervals than the large shift intervals.

For small shift interval $\delta \in [0.7,1.25]$, the ARL-unbiased t_4-chart is still superior than all the other charts in terms of PCI whereas the ARL-biased and unbiasedt_4-charts have almost equal RARL values but lower than the remaining charts. It is of interest to note that the ARL-biased t_r-charts for $r = 1$,2 have less PCI values that their counterpart ARL-unbiased charts for smaller shift interval whereas all ARL-biased t_r-charts perform better than the corresponding ARL-unbiased charts in terms of RARL.

V. Conclusion

In this paper, the Shewhart-type t_r-charts are considered which are easy to use and implement for monitoring the variations in the high-yield processes. This study aims to examine the performance of the ARL-biased and -unbiased t_r-charts in terms of various individual and overall performance criteria such as ARL, MRL, EQL, RARL and PCI. It is found that the ARL-unbiased t_r-charts perform better than the corresponding ARL-biased t_r-charts in the process deterioration case which is more serious case from practical point of view. As far as the performance of the charts over a shifting interval, the ARL-unbiased t_4-chart is better than the other charts in terms of RARL, EQL and PCI over the large and moderate shift intervals, however, it is slightly inferior to the ARL-biased t_4-chart in terms of RARL over the small shift interval. Our study also reveals that for small shift interval $\delta \in [0.7,1.25]$, the ARL-biased t_1- and t_2-charts perform better than their counterparts. Thus, based on performance study, it is recommended that for small shift intervals, the ARL-biased t_r-charts can be preferred to ARL-unbiasedt_r-charts whereas for large and moderate shift intervals, the ARL-unbiased t_r-charts.

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